Notes on real orientations

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Abstract

This is an ongoing project on the real orientations in the context of equivariant and chromatic homotopy theory. By a real orientation, we mean a map from $MU_{\mathbf{R}}$, the complex cobordism spectrum MU equipped with a C_2-action by $\mathbb{E}_\infty$-maps. It turns out

1 Motivation and overview

Equivariant foundations, why we care about real-orientations, what good does it do blah blah.

In the remainder of the section we give an outline of the seminal work by Hahn and Shi [HS20]. Their main result is the following theorem.

Theorem 1 ([HS20], 1.3)**.** *Let k be a perfect field of characteristic 2 and Γ a height-n formal group law over k. Denote by $E := E_{(k,\Gamma)}$ the corresponding Lubin-Tate theory. Let G be a finite subgroup of the Morava stabilizer group that contains the central subgroup C_2. Then there is a G-equivariant map $N_{C_2}^G MU_{\mathbf{R}} \to E$.*

The underlying spectrum of $MU_{\mathbf{R}}$ is MU, with a C_2-action induced by complex conjugation on complex manifolds. The $MU_{\mathbf{R}}$ is sometimes referred to as the *real cobordism spectrum*. Recall that a map of commutative ring spectra $MU \to R$ determines a complex orientation on R. We call a C_2-equivariant map $MU_{\mathbf{R}} \to R$ a *real orientation* on R. The theorem above implies that in particular, when $G = C_2$, there is a real orientation $MU_{\mathbf{R}} \to E_n$ on Morava E-theory of height $n \geq 1$.

The proof can be roughly divided into four steps:

(i) a construction of the non-equivariant, 2-periodic version of Johnson-Wilson theory $E(n)$ as an $\mathbb{A}_\infty$-MUP-algebra, starting from a map $X \to B^2GL_1(MUP)$:

$$\text{Spaces}_{/B^2GL_1(MUP)} \xrightarrow{\Omega} \mathbb{A}_\infty(\text{Spaces}_{/BGL_1(MUP)}) \xrightarrow{Thom} \mathbb{A}_\infty(MUP\text{-modules});$$

after $K(n)$-localizing we obtain $\widehat{E(n)}$, which is a model for the Lubin-Tate theory associated to a height-n formal group law *defined over* $\mathbf{F}_2$;

(ii) equip the above construction with appropriate C_2-action; in particular, if we start with a specific C_2-equivariant map $X \to B^\rho GL_1(MUP)$, we obtain a spectrum $E_{\mathbf{R}}(n)$ with a C_2-action via $\mathbb{A}_\infty$-maps, which has $E(n)$ as its underlying spectrum; after $K(n)$-localizing we get a C_2-equivariant version of $\widehat{E(n)}$;

(iii) show that there is a real orientation $MU_{\mathbf{R}} \to \widehat{E(n)}$ using the Goerss-Hopkins-Miller theorem;

(iv) extend the real orientation above to that on the Lubin-Tate theory associated to a finite height formal group law *defined over any perfect field of characteristic* 2, and apply the HHR norm functor $N_{C_2}^G(-)$, citing results from [HM17] and [Mei18].

2 Units and Thom spectra

In this section we give an overview of the construction of units of a ring spectrum and the Thom spectrum functor, which is the starting point of the construction of the 2-periodic Johnson-Wilson theory $E(n)$.

We first explain the construction of units in a $\mathbb{E}_\infty$-ring spectrum A due to [MQRT77]. By units, we mean a spectrum $gl_1 A$ such that $\Omega^\infty gl_1 A \simeq GL_1 A$, where $GL_1 A$ is the *space of units* defined by the following pullback diagram

$$\begin{array}{ccc} GL_1 A & \longrightarrow & \Omega^\infty A \\ \downarrow & & \downarrow \\ (\pi_0 A)^\times & \longrightarrow & \pi_0 A \end{array}$$

of (unpointed) spaces. Since A is an $\mathbb{E}_\infty$-ring spectrum, $GL_1 A$ is a grouplike $\mathbb{E}_\infty$-space, and since grouplike $\mathbb{E}_\infty$-spaces model connective spectra, we have the adjunction

$$\Sigma^\infty_+ \Omega^\infty : \mathrm{ho}((-1)\text{-connected spectra}) \leftrightarrows \mathrm{ho}(\mathscr{S}[\mathbb{E}_\infty]) : gl_1$$

of categories enriched over the homotopy category. Here $\mathscr{S}[\mathbb{E}_\infty]$ is the category of $\mathbb{E}_\infty$-ring spectra modeled by the model category of C-algebras, for some suitable $\mathbb{E}_\infty$-operad C. To elaborate on the adjunction, we note that there is an adjoint pair $(\Sigma^f, \Omega^\infty)$ between the category of (-1)-connected spectra and $\mathscr{T}[C]^\times$ that descends to an equivalence on the homotopy categories; there is also an obvious adjoint pair $(\Sigma^\infty_+, \Omega^\infty)$ between $\mathscr{T}[C]$ and $\mathscr{S}[C]$. Now on the space level the functor GL_1 is right adjoint to the inclusion $\mathscr{T}[C]^\times \to \mathscr{T}[C]$, so we in fact have a sequence of adjunctions

$$\Sigma^\infty_+ \Omega^\infty : (-1)\text{-connected spectra} \underset{\Omega^\infty}{\overset{\Sigma^f}{\leftrightarrows}} \mathscr{T}[C]^\times \underset{GL_1}{\overset{\mathrm{incl}}{\leftrightarrows}} \mathscr{T}[C] \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\leftrightarrows}} \mathscr{I}[C] : gl_1$$

Note that the above adjunction is a vast generalization of the adjunction

$$\mathbf{Z}[-] : \text{abelian groups} \leftrightarrows \text{commutative rings} : GL_1.$$

For the Thom spectrum functor, there are three different definitions, and they are proven to be equivalent in [ABG+08] using Morita theory, which provides a characterization of continuous, homotopy-colimit preserving functors from $\Sigma^\infty_+ GL_1 R$-modules to R-modules as a certain kind of extension of scalars functor. We first introduce the operadic (or, as described by [ABG+08] it, "neo-classical") definition due to Lewis and May. For technical details, we refer to the original treatment in the Chapter IX of [LMSM86]; see also [MS06], and [MQRT77]. Let $\mathscr{I}_c$ be the category of finite dimensional real inner product spaces and linear isometric isomorphisms, which is symmetric monoidal under direct sum. Define an $\mathscr{I}_c$-space to be a continuous functor from $\mathscr{I}_c$ to spaces. The category of of $\mathscr{I}_c$-spaces inherits the symmetric monoidal structure, and commutative monoids for this category model $\mathbb{E}_\infty$-spaces. Commutative monoids T that satisfy an additional condition is called an $\mathscr{I}_c$-FCP (functor with Cartesian product), which gives rise to an $\mathbb{E}_\infty$-space structured by the linear isometries operad. Examples to keep in mind include O, U, and $Spin$.

Consider the $\mathscr{I}_c$-FCP F given by taking $V \subset \mathbf{R}^\infty$ to the space $F(V)$ of based homotopy self-equivalences of S^V, the one-point compactification of V. Passing over to the colimit under inclusion, we get $F(\mathbf{R}^\infty) = \mathrm{colim}_V F(V)$ which is a model for $GL_1 S$, where S is the sphere spectrum. Furthermore, since F is a monoid, applying the levelwise two-sided bar construction yields a $\mathscr{I}_c$-FCP

$$V \mapsto BF(V) = B(*, F(V), *)$$

that provides a model for BGL_1S, the classifying space of stable spherical fibrations. Now readers can probably guess the construction for the $\mathscr{I}_c$-FCP model for EGL_1S, and the levelwise maps

$$EF(V) = B(*, F(V), S^V) \to B(*, F(V), *) = BF(V)$$

are a *universal* quasifibrations with fiber S^V. Given a map $f : X \to BF(\mathbf{R}^\infty)$, by pulling back along subspaces $BF(V) \to BF(\mathbf{R}^\infty)$ we get a filtration $\{X_V\}_V$ on X. Let $E(X_V) \to X(V)$ be the pullback of the universal quasifibration along $X_V \to BF(V)$.

Definition ([LMSM86]). The V-th space of the *Thom prespectrum* is defined to be the Thom space of $E(X_V) \to X(V)$, that is, $E(X_V)_+ \wedge_{F(V)_+} S^V$, obtained by collapsing the section induced from the basepoint inclusion $* \to S^V$. The spectrification is called the *Thom spectrum* of f.

In fact, any filtration of closed subspaces of X ordered by inclusion gives the same Thom spectrum (though not the same Thom prespectra), so we simply denote the Thom spectrum by MF, ignoring the data of a specific $f : X \to BF(\mathbf{R}^\infty)$. It is proved in [LMSM86] that MF is given by

$$MF = \mathrm{colim}_V \Sigma^{-V} \Sigma^\infty MF(V),$$

and since levelwise $MF(V)$ is modeled by the homotopy quotient $S^V/F(V)$, MF is stably equivalent to $S/F(\mathbf{R}^\infty) = S/GL_1S$.

The classical construction above focuses on the sphere spectrum and spherical fibrations. Works of Ando, Blumberg, Gepner, Hopkins, and Rezk [ABG+08], and May-Sigurdsson [MS06] generalized to construct a Thom spectrum associated to a map of spectra $\zeta : b \to \Sigma gl_1 R$, for R an $\mathbb{E}_\infty$-ring spectrum. For obvious reasons, we shall write bgl_1R for $\Sigma gl_1 R$ and Bgl_1R for $\Omega^\infty bgl_1 R$. There is a map of cofiber sequences

$$\begin{array}{ccccc}
g = \Sigma^{-1} b & \xrightarrow{j = \Sigma^{-1}\zeta} & gl_1 R & = & gl_1 R \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & p & \longrightarrow & egl_1 R \simeq * \\
 & & \downarrow & & \downarrow \\
 & & b & \xrightarrow{\zeta} & bgl_1 R
\end{array},$$

where the left and bottom right squares are homotopy pullback squares.

Definition. The *R-algebra Thom spectrum* of ζ is the $\mathbb{E}_\infty$-R-algebra $M\zeta$ that is the pushout in the following diagram of $\mathbb{E}_\infty$-ring spectra:

$$\begin{array}{ccccc}
\Sigma^\infty_+ \Omega^\infty g & \longrightarrow & \Sigma^\infty_+ \Omega^\infty gl_1 R & \xrightarrow{\epsilon} & R \\
\downarrow & & \downarrow & & \downarrow \\
S = \Sigma^\infty_+ \Omega^\infty * & \longrightarrow & \Sigma^\infty_+ \Omega^\infty p & \longrightarrow & M\zeta
\end{array}.$$

That is, $M\zeta$ is given by the derived smash product $S \wedge^L_{\Sigma^\infty_+ \Omega^\infty g} R$, or $\Sigma^\infty_+ \Omega^\infty p \wedge^L_{\Sigma^\infty_+ \Omega^\infty gl_1 R} R$.

After looping down and viewing ζ as an infinite loop map $f : B \to BGL_1 R$, we see that the Thom spectrum of f is the derived smash product $\Sigma^\infty_+ P \wedge^L_{\Sigma^\infty_+ GL_1 R} R$. When one takes $R = S$ so that the map $B \to BGL_1 S$ classifies a stable spherical fibration, we recover the classical construction in [MQRT77] and [LMSM86].

This works all quite well when R is an $\mathbb{E}_\infty$-ring spectrum. But in the case when R is $\mathbb{A}_\infty$, we encounter difficulties due to the fact that GL_1R is not a topological group but rather only a grouplike $\mathbb{A}_\infty$-space. This means that it is more delicate to forming "principal GL_1R-bundles". The solution, as provided by [ABG+08], is to adapt the operadic smash product so that on a subcategory of $\mathscr{T}[C]$ the monoids are precisely $\mathbb{A}_\infty$-spaces. This is based on the theory of rigidified model of $\mathbb{A}_\infty$-spaces as developed in the thesis of Andrew Blumberg.

Another solution that is cleaner but more abstract is the ∞-categorical approach developed in [ABG+08] and [ABG+14].

For a sufficiently nice map $f : B \to BG$ (here G is a nice $\mathscr{F}_c$-FCP, such as O or U), Lewis proved in the Chapter IX of [LMSM86] that the associated Thom spectrum Mf has an enriched ring structure. In particular, operadic analysis shows that if f is an n-fold loop map, then Mf is an E_n-ring spectrum. Using general categorical arguments, the authors of [ACB19] were able to prove the same results for the generalized R-algebra Thom spectra in the ∞-categorical sense. In particular, in our case at hand, by applying the Thom construction to a loop map $f : X \to B^2GL_1(MUP)$, we obtain an $\mathbb{A}_\infty$-MUP-algebra Mf. In the remainder of the section, we shall explain the construction of this particular f, so that Mf is a model for the 2-periodic Johnson-Wilson theory $E(n)$.

First note that for an $\mathbb{E}_\infty$-ring spectrum R,

$$\pi_n(GL_1(R)) = \begin{cases} \pi_0(R)^\times, & n = 0 \\ \pi_n(R), & n \geq 1 \end{cases}.$$

To produce $E(n)$ from MUP, the crudest intuition, obtained by staring at the respective homotopy groups, is to mod out certain elements $v_{n+1}, v_{n+2}, \cdots$. Luckily, taking the Thom spectrum of a nonzero $\alpha \in \pi_2(MUP) \cong \pi_3(BGL_1(MUP))$ is equivalent to the spectra-level of modding out α:

$$M\alpha \simeq \text{cofib}(\Sigma^2 MUP \xrightarrow{\alpha} MUP) =: MUP/\alpha.$$

This interesting result is Theorem 4.1 from [ACB19]. Given two maps $\alpha_1, \alpha_2 : S^3 \to BGL_1(MUP)$, the infinite loop space structure on $BGL_1(MUP)$ produces a product map

$$(\alpha_1, \alpha_2) : S^3 \times S^3 \to BGL_1(MUP) \times BGL_1(MUP) \to BGL_1(MUP)$$

whose Thom MUP-module is $M\alpha_1 \wedge_{MUP} M\alpha_2 \simeq MUP/(\alpha_1, \alpha_2)$. If we stipulate that a sequence $(\alpha_1, \alpha_2, \cdots)$ is regular in $\pi_*(MUP)$, then

$$\pi_*(MUP/(\alpha_1, \alpha_2, \cdots)) \cong \pi_*(MUP)/(\alpha_1, \alpha_2, \cdots).$$

A key observation is that if each α_i is a loop map, then by [ACB19] the associated Thom MUP-module can be refined to an $\mathbb{A}_\infty$-MUP-algebra.

Proposition 1 ([HS20], 2.2)**.** *Each $\alpha_i : S^3 \to BGL_1(MUP)$ can be given the structure of a loop map.*

Proof. We need to construct a map $\tilde{\alpha}_i : BS^3 \to B^2GL_1(MUP)$ such that $\Omega\tilde{\alpha}_i \simeq \alpha_i$. This is equivalent to proving the map

$$S^4 \hookrightarrow BS^3 \xrightarrow{\tilde{\alpha}_i} B^2GL_1(MUP)$$

is adjoint to the map α_i. This follows from the fact that any map $S^4 \to B^2GL_1(MUP)$ factors through BS^3, since $BS^3 \simeq \mathbf{H}P^\infty$ has an even cell decomposition

$$S^4 = Y_1 \to Y_2 \to \cdots \to BS^3, \quad \text{where} \quad \begin{array}{ccc} S^{4n-1} & \longrightarrow & Y_{n-1} \\ \downarrow & & \downarrow \\ D^{4n} & \longrightarrow & Y_n \end{array}$$

is a pushout diagram. The obstruction to factoring through Y_n lies in $\pi_{4n-1}(B^2GL_1(MUP)) \cong \pi_{4n-3}(MUP)$, which is zero since MUP is even periodic. $\square$

Now it only remains to choose an appropriate regular sequence $(\alpha_1, \alpha_2, \cdots)$ such that the $K(n)$-localization of $MUP/(\alpha_1, \alpha_2, \cdots)$ is precisely the Lubin-Tate theory associated to a formal group law of height n over $\mathbf{F}_2$. This is a simple computation on the level of homotopy groups and is explained in detail in Lemma 2.3 of [HS20]. In short, feeding our handcrafted $B^2GL_1(MUP)$-space

$$(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots) : BS^3 \times BS^3 \times \cdots \to B^2GL_1(MUP)$$

to the machine produces for us the desired Lubin-Tate theory. This construction has the benefit of being able to be upgraded to the equivariant setting, to which we now turn attention.

3 Equipping the C_2-action

In this section we summarize the procedure in Section 3 and 4 of [HS20] where the previous construction will be equipped with an appropriate C_2-action.

Let MonCat_{Lax} be the category of monoidal categories and lax monoidal functors. Remark 4.1.1.7 of [Lur18] describes a C_2-action, called *rev*, on MonCat_{Lax}: if $(\mathcal{C}, \otimes)$ is a monoidal category, then $(\mathcal{C}_{rev}, \otimes_{rev})$ has the same underlying category as $\mathcal{C}$ but the opposite $\otimes$-structure. A homotopy fixed point for *rev* is a monoidal category $(\mathcal{C}, \otimes)$ equipped with a coherent equivalence $\mathcal{C} \simeq \mathcal{C}_{rev}$. Such a $(\mathcal{C}, \otimes)$ is called a category *with involution*, which is different from a homotopy fixed point for the trivial C_2-action on MonCat_{Lax}. There is an induced C_2-action on $\mathbb{A}_\infty$-algebra objects $\mathbb{A}_\infty(\mathcal{C})$ in $\mathcal{C}$, with the property that $\mathbb{A}_\infty(\mathcal{C})$ is equivalent to $\mathbb{A}_\infty(\mathcal{C}_{rev})$, the *opposite algebra*. We referred to this C_2-action on $\mathbb{A}_\infty(\mathcal{C})$ the *op* action, and write $\mathbb{A}^\sigma_\infty(\mathcal{C})$ to emphasize the action.

Now we specialize to our case of interest. Take $\mathcal{C}$ to be the category Spaces of (pointed) spaces. An $\mathbb{A}_\infty$-space X with involution is a homotopy fixed point for the *op* action on $\mathbb{A}^\sigma_\infty(\mathrm{Spaces})$. Then the monoidal overcategory $\mathrm{Spaces}_{/X}$ is also equipped with involution.

Denote the sign representation of C_2 by σ and the 2-dimensional regular representation by ρ. In the C_2-equivariant setting, we are more interested in Ω^σ, based maps from the representation spheres S^σ, instead of the ordinary loop space Ω. There are now two different equivalences

$$\underset{\circlearrowright\, op}{\text{grouplike } \mathbb{A}_\infty\text{-spaces}} \xleftarrow{\ \Omega^\sigma\ } \underset{\circlearrowright\, triv}{\text{connected spaces}} \xrightarrow{\ \Omega\ } \underset{\circlearrowright\, triv}{\text{grouplike } \mathbb{A}_\infty\text{-spaces}} .$$

Therefore, if X is a grouplike $\mathbb{A}_\infty$-space with involution, then there exists a connected space $B^\sigma X$ with trivial C_2-action such that $\Omega^\sigma B^\sigma X \simeq X$. Thus, Ω^σ is a C_2-equivariant functor

$$\Omega^\sigma : \mathrm{Spaces}_{/B^\sigma X} \to \mathbb{A}^\sigma_\infty(\mathrm{Spaces}_{/X}),$$

sharing the same underlying functor as Ω. Then, Hahn and Shi make the observation that the C_2-action on $\Omega^\sigma(X)$ for any C_2-space X is given by precomposing with the the complex conjugation action on S^1 and postcomposing with the C_2-action on X.

In our present case, we want a good C_2-action on MUP. This requires a more geometric interpretation of the periodic complex bordism spectrum MUP. Recall that MUP can be constructed as a homology theory using the Landweber exactness theorem by formally inverting u, an element in degree 2:

$$MUP_*(X) := MU_*[u^{\pm 1}] \otimes_{MU_*} MU_*(X).$$

Alternatively, invoking techniques from Chapter V of [EKMM97], the homotopy ring map that forces $\pi_*(MUP)$ to be $\pi_*(MU)[u^{\pm 1}]$ equivalently produces MUP as a spectrum. In [HS20] the authors describes MUP as arising as the Thom spectrum of the complex J-homomorphism. The complex conjugation by infinite loop maps on $BU \times \mathbf{Z}$ thus gives the desired C_2-action on MUP by E_∞-ring maps. This in turn induces C_2-actions by monoidal functors on the category of MUP-modules and $\mathrm{Spaces}_{/BGL_1(MUP)}$.

Combining the discussion above, we have successfully set up the equivariant machinery, that is, a C_2-equivariant diagram (in the homotopy fixed point category of MonCat_{Lax} with respect to the *rev* action)

$$\mathrm{Spaces}_{/B^\rho GL_1(MUP)} \xrightarrow{\Omega} \mathbb{A}^\sigma_\infty(\mathrm{Spaces}_{/B^\sigma GL_1(MUP)}) \xrightarrow{Thom} \mathbb{A}^\sigma_\infty(MUP\text{-modules}),$$

where B^ρ is our notation for BB^σ. Note that we sometimes write $MU_{\mathbf{R}}P$ for spectrum MUP equipped with the C_2-action. Now what remains is to construct a specific $B^\rho GL_1(MUP)$-space with underlying map of spaces

$$BS^3 \times BS^3 \times \cdots \to B^2GL_1(MUP)$$

just like before, such that feeding it into the machine above produces the C_2-equivariant version of $E(n)$, the 2-periodic Johnson-Wilson theory.

Upon reviewing our construction in the non-equivariant case, we first need to understand the quaternionic projective space $\mathbf{H}P^\infty$ as a model of $B^\sigma S^{\rho+1}$, C_2-equivariantly. This is Proposition 4.2 of [HS20]. Firstly, there is a C_2-even cell decomposition on $\mathbf{H}P^\infty$ due to Mike Hopkins, where $\mathbf{H}P^\infty$ arises as a filtered colimit

$$S^{2\rho} = Y_1 \to Y_2 \to \cdots \to \mathbf{H}P^\infty, \quad \text{where} \quad \begin{array}{ccc} S^{2n\rho-1} & \longrightarrow & Y_{n-1} \\ \downarrow & & \downarrow \\ * & \longrightarrow & Y_n \end{array}$$

is a homotopy pushout square of C_2-spaces. The C_2-action on $\mathbf{H}P^\infty$ is given by conjugation by i:

$$[z_0 : z_1 : \cdots] \mapsto [iz_0i^{-1} : iz_1i^{-1} : \cdots].$$

The expression $i(a + bi + cj + dk)i^{-1} = a + bi - cj - dk$ tells us that the C_2-cells attached are multiples of 2ρ. Under this action, $\mathbf{C}P^\infty$ sits as a fixed point inside $\mathbf{H}P^\infty$. The C_2-equivariant map $S^{2\rho} \to \mathbf{H}P^\infty$ given by the cell decomposition lifts the non-equivariant map $S^4 \to BS^3$, and gives a map $f : S^{\rho+1} \to \Omega^\sigma \mathbf{H}P^\infty$ of C_2-spaces which is an equivalence of underlying non-equivariant spaces. Hahn and Shi show that f is in fact a C_2-equivalence by checking that it is an equivalence on C_2-fixed points. To do that we first write f as the composite of the unit map $S^{\rho+1} \to \Omega^\sigma\Sigma^\sigma S^{\rho+1}$ and Ω^σ applied to $S^{2\rho} \to \mathbf{H}P^\infty$. We then form the following commutative diagram of C_2-fixed point:

$$
\begin{array}{ccccc}
 & & (S^{\rho+1})^{C_2} = S^2 & & \\
 & & \big\downarrow{\scriptstyle 1} & & \\
 & & (\Omega^\sigma \Sigma^\sigma S^{\rho+1})^{C_2} & & \\
 & \swarrow^{2} & & \searrow^{3} & \\
(\Sigma^\sigma S^{\rho+1})^{C_2} & & & & (\Omega^\sigma \mathbf{H}P^\infty)^{C_2} \\
 & \searrow_{4} & & \swarrow_{5} & \\
 & & (\mathbf{H}P^\infty)^{C_2} = \mathbf{C}P^\infty & & \\
 & & \big\downarrow{\scriptstyle 6} & & \\
 & & \mathbf{H}P^\infty & &
\end{array}
\cdot
$$

We shall compute $(\Sigma^\sigma S^{\rho+1})^{C_2}$ and $(\Omega^\sigma \mathbf{H}P^\infty)^{C_2}$ and find them both equivalent to S^2, so that maps 4 and 5 are inclusions into $\mathbf{C}P^\infty$. We further deduce that $2 \circ 1$ is an equivalence, so that by commutativity $3 \circ 1$ is also an equivalence, which is what we want. Indeed, since $\mathbf{C}P^\infty \to \mathbf{H}P^\infty$ is the map $BS^1 \to BS^3$, its fiber $(\Omega^\sigma \mathbf{H}P^\infty)^{C_2}$ is $S^3/S^1 \simeq S^2$. On the other hand, for any C_2-space X, the composite $X \to \Omega^\sigma \Sigma^\sigma X \to \Sigma^\sigma X$, where the second map is obtained by sending $f : S^\sigma \to \Sigma^\sigma X$ to $f(S^0)$, induces an equivalence $X^{C_2} \simeq (\Sigma^\sigma X)^{C_2}$ on C_2-fixed points, which implies that $2 \circ 1$ is an equivalence.

As a corollary of the above discussion, the exact same argument as in Proposition 1 shows that the obstruction of any map $S^{2\rho} \to B^\rho GL_1(MU_{\mathbf{R}}P)$ factoring through $B^\sigma S^{\rho+1}$ lies in

$$\pi_{2n\rho-1}(B^\rho GL_1(MU_{\mathbf{R}}P)) \cong \pi_{(2n-1)\rho-1}(MU_{\mathbf{R}}P),$$

which is zero by the computation in [HK01]. Now, by choosing an analogous regular sequence

$$(\alpha_1, \alpha_2, \cdots) : B^\sigma S^{\rho+1} \times B^\sigma S^{\rho+1} \times \cdots \to B^\rho GL_1(MU_{\mathbf{R}}P)$$

as in the non-equivariant setting and applying the Ω^σ and the Thom spectra functor, we obtain a homotopy fixed point of $\mathbb{A}_\infty^\sigma(MUP\text{-modules})$, which we denote by $E_{\mathbf{R}}(n)$. The underlying $\mathbb{A}_\infty$-ring spectrum of $E_{\mathbf{R}}(n)$ is of course the 2-periodic version of Johnson-Wilson theory $E(n)$, but $E_{\mathbf{R}}(n)$ comes with an involution $E(n) \simeq E(n)^{op}$ by $\mathbb{A}_\infty$-ring maps that lifts the C_2 action on $E(n)$ by complex conjugation.

We are almost ready to prove the main theorem. Denote by $\widehat{E(n)}$ the $K(n)$-localization of $E_{\mathbf{R}}(n)$. We now have a Galois C_2-action by involution on the Morava E-theory associated to a height-n formal group law defined over F_2. However, it is not a priori clear that this action is given by the real orientation, that is, is compatible with the C_2-equivariant map from $MU_{\mathbf{R}}$. We now prove that it is indeed the case, using the Goerss-Hopkins-Miller theorem.

Proposition 2 ([HS20], 1.6)**.** *The spectrum $\widehat{E(n)}$ with its central Galois C_2-action, is real oriented, i.e., receives a C_2-equivariant map from $MU_{\mathbf{R}}$.*

Proof. Let $\mathcal{C}_1$ be the category of all spectra equivalent to $\widehat{E(n)}$ and equivalences between them. Let $\mathcal{C}_2$ (resp. $\mathcal{C}_3$) be the category of $\mathbb{A}_\infty$-ring spectra (resp. $\mathbb{E}_\infty$-ring spectra) with $\widehat{E(n)}$ as underlying spectrum, and equivalences between them. The equivariant sequence of forgetful functors $\mathcal{C}_3 \to \mathcal{C}_2 \to \mathcal{C}_1$ comes from the sequence

$$\mathbb{E}_\infty(\text{Spectra}) \to \mathbb{A}_\infty^\sigma(\text{Spectra}) \to \text{Spectra},$$

where $\mathbb{E}_\infty(\text{Spectra})$ and Spectra are equipped with trivial C_2-action. A map of categories with C_2-action is an equivalence if and only if the underlying non-equivariant functor is an equivalence of non-equivariant categories. By the Goerss-Miller-Hopkins theorem, the map $\mathcal{C}_2 \to \mathcal{C}_1$ is an equivalence; they are equivalent to $B\mathbb{G}_n$, where $\mathbb{G}_n$ is the Morava stabilizer group. Thus, any homotopy fixed-point of $\mathcal{C}_2$ is uniquely lifted to one of $\mathcal{C}_3$, and in particular, the C_2-action on $\widehat{E(n)}$ via $\mathbb{A}_\infty$-involutions has a unique lift to a C_2-action by $\mathbb{E}_\infty$-automorphisms, which is the data of a map $BC_2 \to B\mathbb{G}_n$, or equivalently, $C_2 \to \mathbb{G}_n$. question □

proof of Theorem 1. We need to show that there is a real orientation on the Lubin–Tate theory $E_{(k,\Gamma)}$ associated to any finite height formal group law Γ over *any* perfect field k of characteristic 2, not just $\mathbf{F}_2$. In general, there is a map $E_{(k,\Gamma)} \to E_{(\bar{k},\bar{\Gamma})}$, where $\overline{\Gamma}$ is the pushforward of Γ/k to the algebraic closure $\overline{\Gamma}/\bar{k}$. Since any two height-$n$ formal groups over an algebraically closed field are isomorphic, Γ is isomorphic to the Honda formal group defined over $\mathbf{F}_2$, which by Proposition 2 is real oriented. A computation shows that $\pi_{*\rho}^{C_2} E_{(\bar{k},\bar{\Gamma})}$ is a copy of the non-equivariant homotopy group $\pi_{2*}E_{(\bar{k},\bar{\Gamma})}$, so that $E_{(\bar{k},\bar{\Gamma})}$ is a free $E_{(k,\Gamma)}$-module. Thus, since $\pi_{*\rho-1}^{C_2} E_{(\bar{k},\bar{\Gamma})} = 0$, so is $\pi_{*\rho}^{C_2} E_{(k,\Gamma)}$, i.e., the C_2-equivariant spectrum $E_{(k,\Gamma)}$ is even. Lemma 3.3 from [HM17] says that every such spectrum is real orientable. Finally, if G is a finite subgroup of $\mathbb{G}_n$ containing C_2 as a central subgroup, then we can leverage the Hill-Hopkins-Ravenel norm functor $N_{C_2}^G(-)$ and get a sequence of G-equivariant homotopy ring maps

$$N_{C_2}^G(MU_{\mathbf{R}}) \to N_{C_2}^G(E_{(k,\Gamma)}) \to E_{(k,\Gamma)}.$$

□

4 Applications to computation

References

[ABG+08] g

[ABG+14] f

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