Power operations and HKR character theory

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Abstract

This is a set of notes the author took when reading [Stap].

1 Overview

As a consequence of the seminal theorem by Goerss, Hopkins, and Miller, each Morava E-theory spectrum $E(n)$ admits a unique E_∞-ring structure. This multiplicative structure can be exploited to define a *power operation* $P_m : E^0(X) \to E^0(X \times B\Sigma_m)$ that refines the m-th power map $x \mapsto x^{\times m} : E^0(X) \to E^0(X^m)$ in the following sense:

$$\begin{array}{ccccc} & & E^0(X^m \times_{\Sigma_m} E\Sigma_m) & \xrightarrow{\text{diag}} & {}^0(X \times B\Sigma_m) \\ & \overset{P_m}{\nearrow} & \big\downarrow \text{fib} & & \big\downarrow {*\to B\Sigma_m} \\ E^0(X) & \xrightarrow[x \mapsto x^{\times m}]{} & E^0(X^m) & \xrightarrow[\text{diag}]{} & E^0(X) \end{array}$$

Here Σ_m is the symmetric group on m letters. Power operations are multiplicative but not additive, and it purely comes from the multiplicative structure of $E(n)$. We are particularly interested in the case when $X = BG$ for finite groups G. This is because the Atiyah-Segal completion theorem provides a description of the $K^0(BG)$ in terms of the representation ring of G, and Morava E-theory is a generalization of p-adic K-theory to higher height. Therefore, $E^0(BG)$ is a natural generalization of the representation ring of G. For instance, as we shall see, the Strickland's theorem provides an algebro-geometric description of a quotient of $E^0(B\Sigma_{p^r})$.

On the other hand, relying purely on the additive structure of $E(n)$, Hopkins, Kuhn, and Ravenel defined maps $\chi : E^0(BG) \to \mathrm{Cl}(G, C_0)$ which are analogous to the map in representation theory associating to each representation its character. Here $\mathrm{Cl}(G, C_0)$ is analogous to the ring of class functions on G taking values in a ring C_0. The ring $\mathrm{Cl}(G, C_0)$ comes equipped with a natural action of $GL_n(\mathbf{Z}_p)$, and that Hopkins, Kuhn, and Ravenel proved that χ induces a rational isomorphism

$$\mathbf{Q} \otimes E^0(BG) \cong \mathrm{Cl}(G, C_0)^{GL_n(\mathbf{Z}_p)},$$

which provides a good algebraic description of $E^0(BG)$ modulo torsion.

It's natural to ask if we can extend P_m, via χ, to a power operation on $\mathrm{Cl}(G, C_0)$, that is, can we fill the following commutative diagram:

$$\begin{array}{ccc} E^0(BG) & \xrightarrow{P_m} & E^0(BG \times B\Sigma_m) \\ \chi \big\downarrow & & \big\downarrow \chi \\ \mathrm{Cl}(G, C_0) & \overset{?}{\dashrightarrow} & \mathrm{Cl}(G \times \Sigma_m, C_0) \end{array} \qquad ?$$

It is surprising that P_m and χ should interact since one comes from the multiplicative structure of $E(n)$ and the other from the additive structure. There are three ingredients that goes into the solution of the aforementioned problem:

(i) the ring C_0, which is intimately related to certain moduli problems over the Lubin-Tate space, and the symmetries of those moduli problems;

(ii) the result of Ando, Hopkins, and Strickland, which gives an algebro-geometric description of a special case of P_m in terms of those moduli problems;

(iii) the HKR character theory, which implies that the E-cohomology of finite groups can be detected by the E-cohomology of its abelian subgroups.

2 Moduli problems over the Lubin-Tate space and their symmetries

Let us fix the data of a formal group $\mathbb{F}$ of height n over a perfect field κ of characteristic p.

Recall that a deformation of $\mathbb{F}/\kappa$ to a complete local ring $(R, \mathfrak{m})$ is a triple of data

$$(\mathbb{G}, i, \tau) := (\mathbb{G}/R,\ i : \kappa \hookrightarrow R/\mathfrak{m},\ \tau : \pi^*\mathbb{G} \xrightarrow{\cong} i^*\mathbb{F}),$$

where $\pi : R \to R/\mathfrak{m}$ and τ is an isomorphism of formal groups over $R/\mathfrak{m}$. In other words, $\mathbb{G}$ is a formal group over R such that the pullback of $\mathbb{G}$ over its special fiber $R/\mathfrak{m}$ is isomorphic to the pullback of $\mathbb{F}$ over the same fiber. Diagrammatically:

$$\begin{array}{ccccccc}
\mathbb{G} & \longleftarrow & \pi^*\mathbb{G} & \xrightarrow{\quad\cong\quad} & i^*\mathbb{F} & \longrightarrow & \mathbb{F} \\
\downarrow & & \searrow & & \swarrow & & \downarrow \\
\operatorname{Spec} R & \xleftarrow{\quad\pi\quad} & & \operatorname{Spec} R/\mathfrak{m} & & \xrightarrow{\quad i\quad} & \operatorname{Spec} \kappa
\end{array}\ .$$

A *⋆-isomorphism* between two deformations $(\mathbb{G}, i, \tau)$ and $(\mathbb{G}', i', \tau')$ of $\mathbb{F}/\kappa$ to $(R, \mathfrak{m})$ is $i = i'$ and $\delta : \mathbb{G} \to \mathbb{G}'$ is an isomorphism of formal groups compatible with τ and τ' in the sense that the following diagram commutes:

$$\begin{array}{ccc}
\pi^*\mathbb{G} & \xrightarrow{\ \tau\ } & i^*\mathbb{F} \\
{\scriptstyle \pi^*\delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
\pi^*\mathbb{G}' & \xrightarrow{\ \tau'\ } & i^*\mathbb{F}
\end{array}\ .$$

There is a moduli problem LT : CompLocRings $\to$ Groupoids from the category of complete local rings to the category of groupoids by sending $(R, \mathfrak{m})$ to the groupoid of deformations of $\mathbb{F}/\kappa$ to $(R, \mathfrak{m})$ with ⋆-isomorphisms. To see that LT defines a functor, note that for a map $j : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ of complete local rings, there is an induced map of groupoids $\mathrm{LT}(R, \mathfrak{m}) \to \mathrm{LT}(S, \mathfrak{n})$ defined by sending

$$(G, i, \tau) \mapsto (j^*\mathbb{G},\ \kappa \xrightarrow{i} R/\mathfrak{m} \xrightarrow{j/\mathfrak{m}} S/\mathfrak{n},\ (j/\mathfrak{m})^*\tau)$$

that fits into the following commutative diagram:

$$
\begin{array}{ccccccc}
\mathbb{G} & \longrightarrow & \operatorname{Spec} R & \xleftarrow{\;j\;} & \operatorname{Spec} S & \longleftarrow & j^*\mathbb{G} \\
\uparrow & & \uparrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \pi'} & & \uparrow \\
\pi^*\mathbb{G} & \longrightarrow & \operatorname{Spec} R/\mathfrak{m} & \xleftarrow{\;j/\mathfrak{m}\;} & \operatorname{Spec} S/\mathfrak{n} & \longleftarrow & (\pi')^*j^*\mathbb{G} = (j/\mathfrak{m})^*\mathbb{G} \\
{\scriptstyle \tau}\downarrow{\scriptstyle \cong} & \nearrow & & \searrow{\scriptstyle i} & & & {\scriptstyle \cong}\downarrow{\scriptstyle (j/\mathfrak{m})^*\tau} \\
 & & \mathbb{F} & \longrightarrow & \operatorname{Spec} \kappa & & \\
 & \nearrow & & & & & \\
i^*\mathbb{F} & & & \longleftarrow & & & (j/\mathfrak{m})^*i^*\mathbb{F}
\end{array}
.
$$

It is a classical theorem of Lubin and Tate that the moduli problem LT has a solution: the functor LT is corepresented by a complete local ring (called the Lubin-Tate ring) $\mathcal{O}_{LT}$ noncanonically isomorphic to $W(\kappa)[[u_1, \cdots, u_{n-1}]]$, wher $W(\kappa)$ is the ring of p-typical Witt vectors over κ. Therefore, there is a universal deformation $(\mathbb{G}_u/\mathcal{O}_{LT}, \mathrm{id}_\kappa, \mathrm{id}_\mathbb{F})$ such that for any deformation $(\mathbb{G}, i, \tau)$ over $(R, \mathfrak{m})$ there is a map $j : \mathcal{O}_{LT} \to R$ such that $(j^*\mathbb{G}_u, j/\mathfrak{m}, \mathrm{id}_{(j/\mathfrak{m})^*\mathbb{F}})$ that is $\star$-isomorphic to $(\mathbb{G}, i, \tau)$:

$$
\begin{array}{ccccccc}
\mathbb{G}_u & \longrightarrow & \operatorname{Spec} \mathcal{O}_{LT} & \xleftarrow{\;j\;} & \operatorname{Spec} R & \longleftarrow & j^*\mathbb{G}_u \\
\uparrow & & \uparrow{\scriptstyle \pi} & & \uparrow & & \uparrow \\
\pi^*\mathbb{G}_u & \longrightarrow & \operatorname{Spec} \kappa & \xleftarrow{\;j/\mathfrak{m}\;} & \operatorname{Spec} R/\mathfrak{m} & \longleftarrow & (j/\mathfrak{m})^*\mathbb{G}_u \\
{\scriptstyle \mathrm{id}_\mathbb{F}}\downarrow{\scriptstyle \cong} & \nearrow & & & & & {\scriptstyle \cong}\downarrow{\scriptstyle \mathrm{id}_{(j/\mathfrak{m})^*\mathbb{F}}} \\
\mathbb{F} & & & \longleftarrow & & & (j/\mathfrak{m})^*\mathbb{F}
\end{array}
.
$$

Building on LT, there are a few other moduli problems. Let A be a finite abelian group and $\mathbb{G}$ a formal group over a complete local ring $(R, \mathfrak{m})$. A map $f : A \to \mathbb{G}$ over R is just a map of abelian groups $A \to \mathbb{G}(R)$. Such a map f is called a *level structure* if

$$
f(A) := \operatorname{Spf}\left(\frac{R[[x]]}{\prod_{a \in A} x - f(a)}\right)
$$

is a subgroup scheme of $\mathbb{G}$ and that $\operatorname{rank} A \leq \operatorname{ht} \mathbb{G}$. We define functors

$$
\operatorname{Hom}(A, \mathbb{G}_u),\ \operatorname{Level}(A, \mathbb{G}_u),\ \operatorname{Sub}_{p^k}(\mathbb{G}_u) : \mathrm{CompLocRings} \to \mathrm{Groupoids},
$$

defined by

(i) $\operatorname{Hom}(A, \mathbb{G}_u)(R, \mathfrak{m})$ is the groupoid with objects triples $(f : A \to \mathbb{G}, i, \tau)$ with $\star$-isomorphisms $\delta : \mathbb{G} \to \mathbb{G}'$ that commute with the structure in the sense that

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;} & \mathbb{G} \\
 & {\scriptstyle f'}\searrow & \downarrow{\scriptstyle \delta} \\
 & & \mathbb{G}'
\end{array}
;
$$

(ii) $\operatorname{Level}(A, \mathbb{G}_u)(R, \mathfrak{m})$ is the groupoid with objects triples $(\ell : A \to \mathbb{G}, i, \tau)$ with $\star$-isomorphisms that commute with the level structures $\ell : A \to \mathbb{G}$;

(iii) $\mathrm{Sub}_{p^k}(\mathbb{G}_u)(R, \mathfrak{m})$ is the groupoid with objects triples $(H \subset \mathbb{G}, i, \tau)$, where $H \subset \mathbb{G}$ is a subgroup scheme of order p^k, with $\star$-isomorphisms that send H to itself.

All those moduli problems have nice solutions: they are corepresented by complete local rings $\mathcal{O}_{\mathrm{Hom}(A,\mathbb{G}_u)}$, $\mathcal{O}_{\mathrm{Level}(A,\mathbb{G}_u)}$, and $\mathcal{O}_{\mathrm{Sub}_{p^k}(\mathbb{G}_u)}$ that are finitely generated and free as modules over $\mathcal{O}_{\mathrm{LT}}$. This relies crucially on the Weierstrass preparation theorem, which can be applied precisely because the Lubin-Tate ring $\mathcal{O}_{\mathrm{LT}}$ is a complete local ring. We shall also denote $\mathcal{O}_{\mathrm{LT}}$ by E^0 associated to the Morava E-theory.

Those moduli problems are highly symmetric in the sense that they are acted on by endomorphism groups. Indeed, there is a well-defined left action of $\mathrm{Aut}(A)$ on both $\mathrm{Hom}(A, \mathbb{G}_u)$ and $\mathrm{Level}(A, \mathbb{G}_u)$ given by precomposition with the inverse. The action is via $\mathcal{O}_{\mathrm{LT}}$-algebra maps, since it does not effect $\mathbb{G}$, i, or τ. There is also a right action of the Morava stabilizer group $\mathrm{Aut}(\mathbb{F}/\kappa)$ on the moduli problems given by $(\mathbb{G}, i, \tau) \cdot s = (\mathbb{G}, i, (i^* s)\tau)$.

Let $\mathbb{L} = \mathbf{Z}_p^{\times n}$ and $\mathbb{T} = \mathbb{L}^\vee \cong (\mathbf{Q}_p/\mathbf{Z}_p)^n$, the Pontryagin dual of $\mathbb{L}$. The letter $\mathbb{L}$ is for (p-adic) lattice and $\mathbb{T}$ is for (p-adic) torus. The p^k-torsion points of $\mathbb{T}$ is given by $\mathbb{T}[p^k] \cong (\mathbb{L}/p^k\mathbb{L})^\vee$, and both $\mathbb{T}[p^k]$ and its dual are isomorphic to $(\mathbf{Z}/p^k\mathbf{Z})^n$. Specializing to the case $A = \mathbb{T}[p^k]$, we see that there are left and right actions:

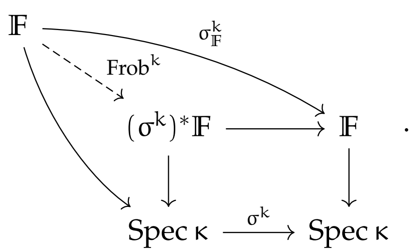
$$\mathrm{Aut}(\mathbb{T}[p^k]) \cong GL_n(\mathbf{Z}/p^k\mathbf{Z}) \curvearrowright \mathrm{Level}(\mathbb{T}[p^k], \mathbb{G}_u) \curvearrowleft \mathrm{Aut}(\mathbb{F}, \kappa).$$

There are functors $\mathrm{Level}(\mathbb{T}[p^k], \mathbb{G}_u) \to \mathrm{Level}(\mathbb{T}[p^{k-1}], \mathbb{G}_u)$ sending a level structure $\mathbb{T}[p^k] \to \mathbb{G}$ to the composite $\mathbb{T}[p^{k-1}] \subset \mathbb{T}[p^k] \to \mathbb{G}$. Let $\mathrm{Level}(\mathbb{T}, \mathbb{G}_u) = \lim_k \mathrm{Level}(\mathbb{T}[p^k], \mathbb{G}_u)$, where the groupoid $\mathrm{Level}(\mathbb{T}, \mathbb{G}_u)(R, \mathfrak{m})$ consists of triples $(\ell : \mathbb{T} \to \mathbb{G}, i, \tau)$ up to compatible $\star$-isomorphisms such that the induced map $\mathbb{T}[p^k] \to \mathbb{G}$ is a level structure for any $k \geq 0$.

Definition. The ring of functions $\mathcal{O}_{\mathrm{Level}(\mathbb{T},\mathbb{G}_u)} = \mathrm{colim}_k\, \mathcal{O}_{\mathrm{Level}(\mathbb{T}[p^k],\mathbb{G}_u)}$ is called the *Drinfeld ring*. The ring $C_0 := \mathbf{Q} \otimes \mathcal{O}_{\mathrm{Level}(\mathbb{T},\mathbb{G}_u)}$.

The key observation is that when taken limits, the moduli problem $\mathrm{Level}(\mathbb{T}, \mathbb{G}_u)$ now picks up extra symmetries in the sense that the left action of $\mathrm{Aut}(\mathbb{T})$ extends to an action of $\mathrm{Isog}(\mathbb{T})$, the monoid of endoisogenies (endomorphisms with finite kernel), which is much larger.

Let σ^k be the k-th Frobenius endomorphism on κ and $\sigma^k_{\mathbb{F}}$ the k-th Frobenius endomorpshim on $\mathbb{F}$. Recall the k-th relative Frobenius $\mathrm{Frob}^k : \mathbb{F} \to \sigma^*\mathbb{F}$, which is obtained by the universal property of pullback in the following diagram:



Let $H \subset \mathbb{G}$ be a subgroup scheme of order p^k. Define the quotient $\mathbb{G}/H$ to be the coequalizer

$$\mathbb{G}/H := \mathrm{coeq}\left(\mathbb{G} \times H \underset{\text{proj.}}{\overset{\text{action}}{\rightrightarrows}} \mathbb{G} \right).$$

There is a canonical way to extend $\mathbb{G}/H$ to a deformation of $\mathbb{F}/\kappa$. Let $q : \mathbb{G} \to \mathbb{G}/H$ be the quotient map and consider the following diagram of formal groups over $R/\mathfrak{m}$:

$$\begin{array}{ccc} \pi^*\mathbb{G} & \xrightarrow{\quad\tau\quad} & i^*\mathbb{F} \\ {\scriptstyle \pi^* q}\big\downarrow & & \big\downarrow{\scriptstyle i^*\mathrm{Frob}^k} \\ \pi^*(\mathbb{G}/H) & \overset{\tau/H}{\dashrightarrow} & i^*(\sigma^k)^*\mathbb{F} \end{array} .$$

Since $\ker(\pi^* q) \subset \pi^*\mathbb{G}$ and $\ker(i^*\mathrm{Frob}^k \circ \tau) \subset \pi^*\mathbb{G}$ have the same order, and that a formal group over a field of characteristic p has a unique subgroup scheme of order p^k for each $k \geq 0$, we must have $\ker(\pi^* q) = \ker(i^*\mathrm{Frob}^k \circ \tau)$. Thus, by the first isomorphism theorem there is a unique isomorphism τ/H that makes the above diagram commute. Therefore, $(\mathbb{G}/H, i \circ \sigma^k, \tau/H)$ forms a deformation of $\mathbb{F}/\kappa$.

Finally, we are able to define the extended action of $\mathrm{Isog}(\mathbb{T})$ on $\mathrm{Level}(\mathbb{T}, \mathbb{G}_u)$. Let $\phi_H \in \mathrm{Isog}(\mathbb{T})$ be an endoisogeny with kernel $H \subset \mathbb{T}$, a finite subgroup of order p^k. For a deformation with level structure $(\ell : \mathbb{T} \to \mathbb{G}, i, \tau)$, set

$$\phi_H \cdot (\ell, i, \tau) = (\mathbb{T} \xrightarrow{\psi_H^{-1}} \mathbb{T}/H \xrightarrow{\ell/H} \mathbb{G}/H,\ i \circ \sigma^k,\ \tau/H),$$

where ψ_H is the unique isomorphism that makes the diagram

$$\begin{array}{ccc} \mathbb{T} & & \\ {\scriptstyle q_H}\big\downarrow & \searrow^{\phi_H} & \\ \mathbb{T}/H & \xrightarrow[\cong]{\psi_H} & \mathbb{T} \end{array}$$

commute. Note that we did not distinguish $H \subset \mathbb{T}$ from $\ell(H) \subset \mathbb{G}$, which is a subgroup scheme of order p^k. Also note that in the case that H is trivial, this recovers the action of $\mathrm{Aut}(\mathbb{T})$ given by precomposition with the inverse.

Hopkins, Kuhn, and Ravenel observed that there is a connection between the Morava E-cohomology of a finite abelian group A and the moduli problems described above, namely, that there is a canonical isomorphisms $E^0(BA) \cong \mathcal{O}_{\mathrm{Hom}(A^\vee, \mathbb{G}_u)}$ of E^0-algebras that is compatible with the action of the stabilizer group $\mathrm{Aut}(\mathbb{F}/\kappa)$.

3 Hopkins-Kuhn-Ravenel character theory

Recall that C_0 is defined to be the rationalization of the Drinfeld ring $\mathcal{O}_{\mathrm{Level}(\mathbb{T}, \mathbb{G}_u)}$. We record a few properties of C_0 without proof:

(i) $C_0 \otimes_{E^0} \mathbb{G}_u[p^k] \cong \mathbb{T}[p^k]$;

(ii) C_0 is an $\mathrm{Aut}(\mathbb{T})$-Galois extension of $\mathbf{Q} \otimes E^0$, where the $\mathrm{Aut}(\mathbb{T})$-action by $\mathbf{Q} \otimes E^0$-algebra maps is inherited from that of $\mathcal{O}_{\mathrm{Level}(\mathbb{T}, \mathbb{G}_u)}$.

Let G be a group. Let us construct the ring $\mathrm{Cl}(G, C_0)$. Consider the action of G on the set $\mathrm{Hom}(\mathbb{L}, G)$ of homomorphisms from $\mathbb{L} \cong \mathbf{Z}_p^{\times n}$ to G by conjugation. Let $\mathrm{Cl}(G, C_0)$ be the ring of conjugation invariant functions on $\mathrm{Hom}(\mathbb{L}, G)$ taking values in C_0, that is,

$$\mathrm{Cl}(G, C_0) = \prod_{\mathrm{Hom}(\mathbb{L}, G)/\sim} C_0,$$

where $\mathrm{Hom}(\mathbb{L}, G)/_\sim$ is the set of G-conjugacy classes of maps from $\mathbb{L}$ to G. Taking a conjugacy class $[\alpha] : \mathbb{L} \to G$, there exists a $k \geq 0$ such that α factors through $\mathbb{L}/p^k\mathbb{L} \to G$. Applying E-cohomology we get a map $E^0(\alpha) : E^0(BG) \to E^0(\mathbb{L}/p^k\mathbb{L})$. Since $\mathbb{L}/p^k\mathbb{L}$ is a finite abelian group and $(\mathbb{L}/p^k\mathbb{L})^\vee = \mathbb{T}[p^k]$, we have a composite of E^0-algebra maps

$$\chi_{[\alpha]} : E^0(BG) \xrightarrow{E^0(\alpha)} E^0(\mathbb{L}/p^k\mathbb{L}) \cong \mathcal{O}_{\mathrm{Hom}(\mathbb{T}[p^k],\mathbb{G}_u)} \to \mathcal{O}_{\mathrm{Level}(\mathbb{T}[p^k]\mathbb{G}_u)} \to \mathcal{O}_{\mathrm{Level}(\mathbb{T},\mathbb{G}_u)} \to C_0$$

coming from the forgetful functor on the moduli problems. Assembling these maps together for different choices of $[\alpha] \in \mathrm{Hom}(\mathbb{L}, G)/_\sim$, we arrive at the definition of the HKR character map $\chi : E^0(BG) \to \mathrm{Cl}(G, C_0)$. Note that when G is finite abelian, $\chi_{[\alpha]}$ is a map on the moduli problem level $\mathrm{Level}(\mathbb{T}, \mathbb{G}_u) \to \mathrm{Hom}(G^\vee, \mathbb{G}_u)$ sending a deformation $(\ell : \mathbb{T} \to \mathbb{G}, i, \tau)$ to $(\ell \circ E^0(\alpha), i, \tau)$.

It is easily checked that χ is $\mathrm{Aut}(\mathbb{F}/\kappa)$-equivariant for the canonical action of the stabilizer group on $E^0(BG)$ and the diagonal action on $\mathrm{Cl}(G, C_0)$. Moreover, there is a left action of $\mathrm{Aut}(\mathbb{T})$ on $\mathrm{Hom}(\mathbb{L}, G)/_\sim$ given by precomposition with the Pontryagin dual. Combining this with the right action of $\mathrm{Aut}(\mathbb{T})$ on C_0 inherited from that of $\mathcal{O}_{\mathrm{Level}(\mathbb{T},\mathbb{G}_u)}$, we obtain an action of $\mathrm{Aut}(\mathbb{T})$ on $\mathrm{Cl}(G, C_0)$ as follows: given $\phi \in \mathrm{Aut}(\mathbb{T})$, $f \in \mathrm{Cl}(G, C_0)$, and $[\alpha] \in \mathrm{Hom}(\mathbb{L}, G)/_\sim$,

$$(\phi \cdot f)([\alpha]) = (f([\alpha\phi^\vee])) \cdot \phi.$$

The base change $C_0 \otimes \chi : C_0 \otimes_{E^0} E^0(BG) \to \mathrm{Cl}(G, C_0)$ of χ to C_0 is $\mathrm{Aut}(\mathbb{T})$-equivariant, where the action of $\mathrm{Aut}(\mathbb{T})$ on the domain is given by the $\mathrm{Aut}(\mathbb{T})$-action on the left tensor factor.

Theorem (HKR)**.** *The character map* χ *induces an isomorphism* $C_0 \otimes \chi : C_0 \otimes_{E^0} E^0(BG) \cong \mathrm{Cl}(G, C_0)$.

Note that by the second property of C_0, we get an isomorphism $\mathbf{Q} \otimes E^0(BG) \cong \mathrm{Cl}(G, C_0)^{\mathrm{Aut}(\mathbb{T})}$ on the $\mathrm{Aut}(\mathbb{T})$-invariants.

4 Power operations and their moduli problem description

Let E be a E_∞-ring spectrum (for example, and X be a space. Let $f \in \Sigma^\infty_+ X \to E$ be an element of $E^0(X)$. The symmetric group Σ_m on m letters acts diagonally on the m-th smash power of f. Taking the homotopy Σ_m-coinvariants (or alternatively, forming the Borel construction) we obtain

$$\begin{array}{ccccccc} \Sigma^\infty_+(X^{\times m}_{h\Sigma_m}) & \longrightarrow & (\Sigma^\infty_+ X)^{\wedge m}_{h\Sigma_m} & \xrightarrow{f^{\wedge m}} & E^{\wedge m}_{h\Sigma_m} & \longrightarrow & E \\ || & & || & & & & \\ \Sigma^\infty_+(E\Sigma_m \times_{\Sigma_m} X^{\times m}) & & (E\Sigma_m)_+ \wedge_{\Sigma_m} (\Sigma^\infty_+ X)^{\wedge m} & & & & \end{array},$$

where the last map is given by the multiplicative structure on E. This map is an element in $E^0(E\Sigma_m \times_{\Sigma_m} X^{\times m})$, so we have produced a map

$$\mathbb{P}_m : E^0(X) \to E^0(E\Sigma_m \times_{\Sigma_m} X^{\times m})$$

which we call the m-th *total power operation*. Since Σ_m acts trivially on the image of the diagonal map $X \to X^{\times m}$, we have an inclusion $X \times B\Sigma_m \to X^{\times m} \times_{\Sigma_m} E\Sigma_m$. Restricting $\mathbb{P}_m$ along this inclusion gives the m-th *power operation*

$$P_m : E^0(X) \xrightarrow{\mathbb{P}_m} E^0(E\Sigma_m \times_{\Sigma_m} X^{\times m}) \to E^0(X \times B\Sigma_m).$$

We are particularly interested in the case when E is Morava E-theory (which has a nice E_∞-ring strucutre due to Goerss-Hopkins-Miller) and $X = BG$ for finite group G. Strickland proved that $E^0(B\Sigma_m)$ is a free E^0-module, so that by the Künneth isomorphism we have

$$P_m : E^0(BG) \to E^0(BG) \otimes_{E^0} E^0(B\Sigma_m).$$

Let $\iota : * \to B\Sigma_M$ be the basepoint inclusion. Then

$$E^0(BG) \xrightarrow{P_m} E^0(BG) \otimes_{E^0} E^0(B\Sigma_m) \xrightarrow{\mathrm{id}\otimes E^0(\iota)} E^0(BG)$$

recovers the m-th power map $x \mapsto x^{\times m}$ on $E^0(BG)$. Both $\mathbb{P}_m$ and P_m are multiplicative but not additive. The failure of additivity is controlled by the theory of transfer maps.

Let $Y \to X$ be an n-fold cover and E a cohomology theory. A transfer map is a map $E^*(Y) \to E^*(X)$ such that precomposing with the restriction $E^*(X) \to E^*(Y)$ is the multiplication-by-n-map on $E^*(X)$. Explicitly, let $P \to X$ be the associated principal Σ_m-bundle of $Y \to X$. We obtain a map

$$t : X \cong P/\Sigma_m \cong P \times_{\Sigma_m} E\Sigma_m \to Y^m \times_{\Sigma_m} E\Sigma_m$$

where latter map is induced by $P \subset Y^m$. Precomposing $E^*(t)$ with the m-th total power operation on $E^*(Y)$ yields the transfer map

$$\mathrm{Tr} : E^*(Y) \xrightarrow{\mathbf{P}_m} E^*(Y^m \times_{\Sigma_m} E\Sigma_m) \xrightarrow{E^*(t)} E^*(X)$$

of $E^*(X)$-modules. Power operations can be recovered from transfer maps and vice versa. Now suppose H is a subgroup of the finite group G. Then $EG \to EG/H \cong EG \times_G (G/H)$ is the universal principal H-bundle, so that we have $BH \cong EG/H$. The induced map $BH \to BG \cong EG/G$ is a G/H-bundle. Specializing to our case at hand, the inclusion $G^m \rtimes (\Sigma_i \times \Sigma_{m-i}) \subset G^m \rtimes \Sigma_m$ induces a finite covering map

$$E(\Sigma_i \times \Sigma_{m-i}) \times_{(\Sigma_i \times \Sigma_{m-i})} (BG)^{\times m} \to E\Sigma_m \times_{\Sigma_m} (BG)^{\times m}$$

which induces a transfer map

$$\mathrm{Tr}_E : E^0\left(E(\Sigma_i \times \Sigma_{m-i}) \times_{(\Sigma_i \times \Sigma_{m-i})} (BG)^{\times m}\right) \to E^0\left(E\Sigma_m \times_{\Sigma_m} (BG)^{\times m}\right)$$

of $E^0(E\Sigma_m \times_{\Sigma_m} (BG)^{\times m})$-modules. Summing over i gives a $E^0(E\Sigma_m \times_{\Sigma_m} (BG)^{\times m})$-algebra map $\oplus_i \mathrm{Tr}_E$, and we denote the image ideal of $\oplus_i \mathrm{Tr}_E$ in $E^0(E\Sigma_m \times_{\Sigma_m} (BG)^{\times m})$ by $\mathbb{I}_{\mathrm{Tr}}$. When G is trivial we denote $\mathbb{I}_{\mathrm{Tr}}$ by $I_{\mathrm{Tr}} \subset E^0(B\Sigma_m)$. These ideals measure the obstruction to additivity of $\mathbb{P}_m$ and P_m in the sense that they are the smallest ideal such that the quotients

$$\mathbb{P}_m/\mathbb{I}_{\mathrm{Tr}} : E^0(BG) \to E^0(E\Sigma_m \times_{\Sigma_m} (BG)^{\times m})/\mathbb{I}_{\mathrm{Tr}}$$

and

$$P_m/I_{\mathrm{Tr}} : E^0(BG) \to E^0(BG) \otimes_{E^0} E^0(B\Sigma_m)/I_{\mathrm{Tr}}$$

are ring homomorphisms.

The transfer map $\mathrm{Tr}_E : E^0(BH) \to E^0(BG)$ can be lifted, via the HKR character map, to a transfer map Tr_{C_0} on the generalized class functions in the sense that the following diagram commutes:

$$\begin{array}{ccc} E^0(BH) & \xrightarrow{\mathrm{Tr}_E} & E^0(BG) \\ \chi \downarrow & & \downarrow \chi \\ \mathrm{Cl}(H, C_0) & \overset{\mathrm{Tr}_{C_0}}{\dashrightarrow} & \mathrm{Cl}(G, C_0) \end{array} .$$

The map Tr_{C_0} defined by Hopkins, Kuhn, and Ravenel is as follows. Given $f \in \mathrm{Cl}(H, C_0)$ and $[\alpha] \in \mathrm{Hom}(\mathbb{L}, G)/_\sim$, there is an action of the image $\mathrm{Im}(\alpha)$ on the cosets G/H inherited from the G-action on G/H. Then $\mathrm{Im}(g^{-1}\alpha g) \subset H$ if and only if $gH \in (G/H)^{\mathrm{Im}(\alpha)}$, the invariants of the $\mathrm{Im}(\alpha)$-action. Define $\mathrm{Tr}_{C_0} : \mathrm{Cl}(H, C_0) \to \mathrm{Cl}(G, C_0)$ by

$$\mathrm{Tr}_{C_0}(f)([\alpha]) = \sum_{gH \in (G/H)^{\mathrm{Im}(\alpha)}} f([g^{-1}\alpha g]).$$

Analogously, there is a transfer ideal $J_{\mathrm{Tr}} \subset \mathrm{Cl}(\Sigma_m, C_0)$ given by the image of the sum of Tr_{C_0} : $\mathrm{Cl}(\Sigma_i \times \Sigma_{m-i}, C_0) \to \mathrm{Cl}(\Sigma_m, C_0)$ over i, which has a nice description given by

$$\mathrm{Cl}(\Sigma_m, C_0)/J_{\mathrm{Tr}} \cong \prod_{\mathrm{Sub}_m(\mathbb{T})} C_0.$$

Results of Ando, Hopkins, and Strickland allows us to build connections between power operations and the moduli problems discussed before. Since E is a p-local theory, $I_{\mathrm{Tr}} = E^0(B\Sigma_m)$ unless m is a p-th power. The following theorem gives an algebro-geometric interpretation of I_{Tr}.

Theorem (Strickland)**.** *There is a canonical isomorphism* $E^0(B\Sigma_{p^k})/I_{\mathrm{Tr}} \cong \mathcal{O}_{\mathrm{Sub}_{p^k}(\mathbb{G}_u)}$ *of* E^0*-algebras.*

A remark is that Strickland's original proof is quite technical. Schlank and Stapleton [SS15] developed an alternative approach using the so-called transchromatic character map.

Recall that $E^0(BA)$ also has a moduli interpretation given by $E^0(BA) \cong \mathcal{O}_{\mathrm{Hom}(A^\vee, \mathbb{G}_u)}$, where A is finite abelian. Now both the domain and codomain of the additive power operation

$$P_{p^k}/I_{\mathrm{Tr}} : E^0(BA) \to E^0(BA) \otimes_{E^0} E^0(B\Sigma_{p^k})/I_{\mathrm{Tr}}$$

yields a moduli problem description, which is the content of the following theorem.

Theorem (Ando-Hopkins-Strickland)**.** *The additive power operation* P_{p^k}/I_{Tr} *is the ring of functions on the map of moduli problems*

$$\mathrm{Sub}_{p^k}(\mathbb{G}_u) \otimes_{\mathrm{LT}} \mathrm{Hom}(A^\vee, \mathbb{G}_u) \to \mathrm{Hom}(A^\vee, \mathbb{G}_u)$$

that, when applied to a complete local ring R*, sends*

$$(H \subset \mathbb{G}, A^\vee \to \mathbb{G}, i, \tau) \mapsto (A^\vee \to \mathbb{G} \to \mathbb{G}/H, i \circ \sigma^k, \tau/H),$$

where H *is a subgroup scheme of order* p^k*, by recalling that* $\mathbb{G}/H$ *is a deformation in a canonical way.*

5 Power operations on the ring of generalized class functions

Finally, we are ready to construct power operations on the ring of generalized class functions that is compatible with power operations on Morava E-theory via the HKR character maps.

Let $\mathrm{Sub}(\mathbb{T})$ be the set of finite groups of $\mathbb{T}$ and $\pi : \mathrm{Isog}(\mathbb{T}) \to \mathrm{Sub}(\mathbb{T})$ be the projection sending an endoisogeny of $\mathbb{T}$ to its kernel. Choose a section $\phi : \mathrm{Sub}(\mathbb{T}) \to \mathrm{Isog}(\mathbb{T})$. For $H \in \mathrm{Sub}(\mathbb{T})$ we denote the corresponding endoisogeny $\phi(H)$ by ϕ_H. We shall produce a map

$$P_m^{\phi} : \mathrm{Cl}(G, C_0) \to \mathrm{Cl}(G \times \Sigma_m, C_0).$$

A class $[\mathbb{L} \to G \times \Sigma_m] \in \mathrm{Hom}(\mathbb{L}, G \times \Sigma_m)/_{\sim}$ is represented by a class $[\alpha] \in \mathrm{Hom}(\mathbb{L}, G)/_{\sim}$ together with a direct sum $\oplus_i H_i$ of subgroups $H_i \subset \mathbb{T}$ with $\sum_i |H_i| = m$. This is because of a bijection

$$\mathrm{Hom}(\mathbb{L}, \Sigma_m)/_{\sim} \cong \{\oplus_i H_i \mid H_i \subset \mathbb{T} \text{ and } \sum_i |H_i| = m\}.$$

Given $f \in \mathrm{Cl}(G, C_0)$, define P_m^{ϕ} by

$$P_m^{\phi}(f)([\alpha], \oplus_i H_i) = \prod_i f([\alpha \circ \phi_{H_i}^{\vee}]) \cdot \phi_{H_i},$$

where the last bit is the right action of $\mathrm{Isog}(\mathbb{T})$ on $C_0 = \mathbf{Q} \otimes \mathcal{O}_{\mathrm{Level}(\mathbb{T}, \mathbb{G}_u)}$.

The solution depends on the choice of ϕ. This is because C_0 is an $\mathrm{Aut}(\mathbb{T})$-extension of $\mathbf{Q} \otimes E^0$. Therefore, the choice disappears after taking $\mathrm{Aut}(\mathbb{T})$-invariants. But first, we need to describe the action of $\mathrm{Aut}(\mathbb{T})$ on $\mathrm{Cl}(G \times \Sigma_m, C_0)$. The action of $\gamma \in \mathrm{Aut}(\mathbb{T})$ on the $[\alpha] \in \mathrm{Hom}(\mathbb{L}, G)/_{\sim}$ bit is given by the usual precomposition, and on $\oplus_i H_i$ is given by $\gamma \cdot \oplus_i H_i = \oplus_i \gamma H_i$. It can be checked that P_m^{ϕ} is $\mathrm{Aut}(\mathbb{T})$-equivariant, and the resulting map

$$\mathrm{Cl}(G, C_0)^{\mathrm{Aut}(\mathbb{T})} \to \mathrm{Cl}(G \times \Sigma_m, C_0)^{\mathrm{Aut}(\mathbb{T})}$$

is independent of the choice of ϕ.

The diagonal action by the Morava stabilizer group $\mathrm{Aut}(\mathbb{F}/\kappa)$ on the generalized class functions commutes with P_m^{ϕ} in the sense that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Cl}(G, C_0) & \xrightarrow{P_m^{\phi}} & \mathrm{Cl}(G \times \Sigma_m, C_0) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ \mathrm{Cl}(G, C_0) & \xrightarrow{P_m^{\phi}} & \mathrm{Cl}(G \times \Sigma_m, C_0) \end{array}$$

for any $s \in \mathrm{Aut}(\mathbb{F}/\kappa)$.

Final remark: one might think about extending the solution for total power operations $\mathbb{P}_m$. This is done in [BS17], but one needs a generalization of Strickland's theorem (which is in [SS15]) and Ando-Hopkins-Strickland (in [BS17]) for the additive total power operations $\mathbb{P}_m/\mathbb{I}_{\mathrm{Tr}}$.

References

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