Kähler manifolds and Hodge theory

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Abstract

These notes were written as the final project for a differential topology course taught by Prof. Eduard Looijenga in the winter of 2023. Familiarity with basic notions of complex analysis and differentiable manifolds is assumed. References include Griffiths-Harris [1], Huybrechts [2], and Voisin [3].

1 Complex manifolds and Kähler manifolds

1.1 Complex manifolds

Recall that a smooth structure on a differentiable manifold is defined as an equivalence class of smooth atlases.

Definition. Let X be a differentiable manifold of dimension $2n$. A *holomorphic atlas* for X is an open cover $\{U_i\}_i$ of X with diffeomorphisms $\kappa_i : U_i \to \kappa_i(U_i) \subset \mathbf{C}^n$ such that the transition maps

$$\kappa_j \circ \kappa_i^{-1} : \kappa_i(U_i \cap U_j) \to \kappa_j(U_i \cap U_j)$$

are holomorphic. A *complex structure* on X is an equivalence class of holomorphic atlases.

Holomorphic functions on a complex manifold and holomorphic maps between complex manifolds are defined analogously to their smooth counterparts. However, complex manifolds are fundamentally different from differentiable ones. For instance, an easy application of the maximum modulus principle from complex analysis shows that there are no non-constant holomorphic functions on a compact connected complex manifold.

There is also the notion of a *holomorphic vector bundle* of rank r over a complex manifold X, consisting of a holomorphic map $\pi : E \to X$ together with local trivializations $\rho_i : \rho_i^{-1}(U_i) \cong U_i \times \mathbf{C}^r$ such that the transition matrices $\rho_j \circ \rho_i^{-1}$ have holomorphic coefficients.

Example (Level sets)**.** This is similar to the real case. Let $f : \mathbf{C}^n \to \mathbf{C}$ be a holomorphic function and let $0 \in \mathbf{C}$ be a regular value for f. Then a complex version of the implicit function theorem shows that $Z(f) = f^{-1}(0)$ is a complex manifold of dimension $n-1$. In fact, this can be generalized to *complete intersections*: if 0 is a regular value for holomorphic functions $(f_1, \cdots, f_m) : \mathbf{C}^n \to \mathbf{C}^m$, then $Z(f_1) \cap \cdots \cap Z(f_m)$ is a complex manifold of dimension $n - m$.

Example (Complex tori)**.** Let V be an n-dimensional complex vector space and $\Gamma \subset V$ a free abelian, discrete subgroup of order $2n$ (so that Γ is freely generated by an $\mathbf{R}$-basis of V), then $X = V/\Gamma$ is a complex manifold. In the case of $V = \mathbf{C}^n$ and $\Gamma = \mathbf{Z}^{2n}$ we get complex tori. In the one-dimensional case, $\mathbf{C}/\Gamma$ is an elliptic curve. In fact, consider the action of $\mathrm{SL}_2(\mathbf{Z})/\{\pm 1\}$ on the upper half-plane $\mathcal{H} = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z \mapsto \frac{az+b}{cz+d}.$$

Any subgroup $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})/\{\pm 1\}$ that acts freely on $\mathcal{H}$ gives rise to a connected 1-dimensional complex manifold $\mathcal{H}/\Gamma$, which is called a *Riemann surface*.

The construction in the previous example can be generalized as follows. Let G be a complex Lie group that acts properly and freely on a topological space X. Then the quotient space X/G has a natural complex manifold structure, with holomorphic quotient map $X \to X/G$.

Example (Hopf manifolds)**.** Consider the free action of the free abelian group $\mathbf{Z}$ on $\mathbf{C}^n \setminus \{0\}$ by $(z_1, \cdots, z_n) \mapsto (\lambda^k z_1, \cdots, \lambda^k z_n)$ for some $k \in \mathbf{Z}$. The quotient manifold $(\mathbf{C}^n \setminus \{0\})/\mathbf{Z}$, which is diffeomorphic to $S^1 \times S^{2n-1}$, is called a Hopf manifold.

We now explain another characterization of complex manifolds. To do that, we need some linear algebra. In the following, let V be a finite-dimensional real vector space.

Definition. An *almost complex structure* on V is a linear endomorphism $I : V \to V$ with $I^2 = -\mathrm{Id}$.

If V is a complex vector space, then $v \mapsto iv$ defines an almost complex structure on the underlying real vector space of V. Conversely, if V admits an almost complex structure I, then V is a complex vector space via $(a+bi) \cdot v = av + bI(v)$. Therefore, for vector spaces, complex structures and almost complex structures are equivalent notions.

Let $V_{\mathbf{C}} := V \otimes_{\mathbf{R}} \mathbf{C}$ be the complexification of a real vector space V. An almost complex structure on V extends $\mathbf{C}$-linearly to an endomorphism of $V_{\mathbf{C}}$, which is again denoted by I. Clearly, the only eigenvalues of I on $V_{\mathbf{C}}$ are $\pm i$. Denote by $V^{1,0}$ and $V^{0,1}$ the eigenspaces of $\pm i$:

$$V^{1,0} = \{v \in V_{\mathbf{C}} : I(v) = iv\}, \quad V^{0,1} = \{v \in V_{\mathbf{C}} : I(v) = -iv\}.$$

Then there is a decomposition $V_{\mathbf{C}} = V^{1,0} \oplus V^{0,1}$. Indeed, since $V^{1,0} \cap V^{0,1} = \emptyset$, the natural map $V^{1,0} \oplus V^{0,1} \to V_{\mathbf{C}}$ is injective, whose inverse is given by $v \mapsto (v - iI(v))/2 \oplus (v + iI(v))/2$. The dual space $V^\vee$ of V has a induced almost complex structure given by $I(f)(v) = f(I(v))$, so that we have a similar eigenspace decomposition $(V^\vee)^{1,0}$ and $(V^\vee)^{0,1}$. If we consider the exterior algebra $\wedge^* V_{\mathbf{C}} = \oplus_{k \geq 0} \wedge^k V_{\mathbf{C}}$, then by looking at the basis we have a decomposition

$$\wedge^k V_{\mathbf{C}} = \oplus_{p+q=k} \wedge^{p,q} V, \quad \text{where} \quad \wedge^{p,q} V := \wedge^p V^{1,0} \otimes_{\mathbf{C}} \wedge^q V^{0,1}.$$

Now we apply this machinery to manifolds.

Definition. An *almost complex structure* on a differentiable manifold X is a vector bundle endomorphism $I : TX \to TX$ satisfying $I^2 = \mathrm{Id}$, where TX is the (real) tangent bundle of X.

Clearly, the dimension of an almost complex manifold must be even.

Unlike vector spaces however, the situation here is a bit more complicated. Every complex manifold admits a natural almost complex structure. Indeed, let $\{\partial/\partial x_1, \cdots, \partial/\partial x_n, \partial/\partial y_1, \cdots, \partial/\partial y_n\}$ be a basis for T_pX. Then $\partial/\partial x_i \mapsto \partial/\partial y_i$ and $\partial/\partial y_i \mapsto -\partial/\partial x_i$ defines an almost complex structure for T_pX. But the converse is false. That is why it is called an "almost" complex structure. To remedy the situation, we need the notion of integrability.

Let X be an almost complex manifold. Let $T_{\mathbf{C}}X$ be the complexification of TX, and $\Omega_{\mathbf{C}}X$ the dual of $T_{\mathbf{C}}X$. Then we have decompositions $T_{\mathbf{C}}X = T^{1,0}X \oplus T^{0,1}X$ and $\Omega^{1,0}X \oplus \Omega^{0,1}X$. To ease the notation, let us denote $\wedge^k \Omega_{\mathbf{C}}X$ and $\wedge^{p,q}\Omega(X)$ by $\Omega^k_{\mathbf{C}}(X)$ and $\Omega^{p,q}(X)$, respectively.

Definition. Let $\Pi^{p,q} : \Omega^k_{\mathbf{C}}(X) \to \Omega^{p,q}(X)$ be the projection operator. Let $d : \Omega^k_{\mathbf{C}}(X) \to \Omega^{k+1}_{\mathbf{C}}(X)$ be the **C**-linear extension of the usual exterior differential. Then we define

$$\partial := \Pi^{p+1,q} \circ d : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X) \quad \text{and} \quad \bar{\partial} := \Pi^{p,q+1} \circ d : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X).$$

Note that ∂ and $\bar{\partial}$ satisfy the Leibniz rule inherited from that of d.

Proposition-Definition 1. *An almost complex structure I on a manifold X is* integrable *if the following equivalent conditions hold:*

(i) $d = \partial + \bar{\partial}$; *(ii) on* $\Omega^{p,q}_X$ *one has* $\Pi^{0,2} \circ d = 0$;

(iii) $[T^{0,1}X, T^{0,1}X] \subset T^{0,1}X$; *(iv)* $\partial^2 = \bar{\partial}^2 = 0$ *and* $\partial\bar{\partial} = -\bar{\partial}\partial$.

Theorem 2 (Newlander-Nierenberg)**.** *Any integrable almost complex structure is induced by a unique complex structure. Therefore, complex manifolds are precisely differentiable manifolds with an integrable almost complex structure.*

For a proof of these integrability criterion, see [2] and [3].

Definition. Let X be a complex manifold. We define the (p,q)-*Dolbeault cohomology* to be

$$H^{p,q}(X) := H^q(\Omega^{p,\bullet}(X), \bar{\partial}) = \ker(\bar{\partial}) / \operatorname{Im}(\bar{\partial}).$$

1.2 Kähler manifolds

Kähler manifolds form an important class of complex manifolds. To be able to define what Kähler manifolds are, we once again go back to linear algebra.

Let V be a finite-dimensional real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$, which is a positive-definite symmetric bilinear form. Let I be an almost complex structure on V.

Definition. The inner product $\langle \cdot, \cdot \rangle$ is *compatible* with an almost complex structure I on V if $\langle I(v), I(w) \rangle = \langle v, w \rangle$ for any $v, w \in V$. The *fundamental form* is defined to be

$$\omega := -\langle \cdot, I(\cdot) \rangle = \langle I(\cdot), \cdot \rangle.$$

It is easily verified that ω is real and of type $(1,1)$, i.e., $\omega \in \wedge^2 V^\vee \cap \wedge^{1,1} V^\vee$. Note that two of the three structure $\{\langle \cdot, \cdot \rangle, I, \omega\}$ determine the remaining one. These structures define a Hermitian form $(\cdot, \cdot) := \langle \cdot, \cdot \rangle - i\omega$ on (V, I). Let $\langle \cdot, \cdot \rangle_{\mathbf{C}}$ be the natural Hermitian extension of $\langle \cdot, \cdot \rangle$ to the complexification $V_{\mathbf{C}}$, that is, $\langle \lambda v, \mu w \rangle_{\mathbf{C}} = \lambda \bar{\mu} \langle v, w \rangle$. Then there is an isomorphism $(V, I) \cong (V^{1,0}, i)$ with $(\cdot, \cdot)/2 = \langle \cdot, \cdot \rangle_{\mathbf{C}}|_{V^{1,0}}$ given by $v \mapsto (v - iI(v))/2$.

Let V be an n-dimensional complex vector space. Let $\{x_i, y_i = I(x_i)\}$ be an **R**-basis for V and $\{x^i, y^i\}$ a dual basis for $V^\vee$. Let $\{z_i = (x_i - I(x_i))/2\}$ and $\{\bar{z}_i = (x_i + iI(x_i))/2\}$ be bases for $V^{1,0}$ and $V^{0,1}$, respectively, and let $\{z^i = x^i + iy^i\}$ and $\{\bar{z}^i = x^i - iy^i\}$ be their dual bases.

Proposition 3. *Suppose the Hermitian form $\langle \cdot, \cdot \rangle_{\mathbf{C}}$ on $V^{1,0}$ with respect to this basis is given by a Hermitian matrix $(h_{ij})/2$. Then the fundamental form is of the form*

$$\omega = \frac{i}{2} \sum_{i,j=1}^{n} h_{ij} z^i \wedge \bar{z}^j.$$

Proof. By definition we have

$$\left\langle \sum_{i=1}^n a_i z_i, \sum_{j=1}^n b_j z_j \right\rangle_{\mathbf{C}} = \frac{1}{2} \sum_{i,j=1}^n h_{ij} a_i \bar{b}_j.$$

Now the correspondence $(\cdot,\cdot) = 2\langle\cdot,\cdot\rangle_{\mathbf{C}}|_{V^{1,0}}$ gives $(x_i, x_j) = h_{ij}$. Since $(\cdot,\cdot)$ is Hermitian on (V, I), we have $(x_i, y_j) = -ih_{ij}$ and $(y_i, y_j) = h_{ij}$. Since $(\cdot,\cdot) = \langle\cdot,\cdot\rangle - i\omega$ by definition, we have $\omega = i\,\mathrm{Im}(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle = \mathrm{Re}(\cdot,\cdot)$. Therefore we have $\omega(x_i, x_j) = \omega(y_i, y_j) = -\mathrm{Im}(h_{ij}) = -\langle x_i, y_j\rangle$ and $\omega(x_i, y_j) = \mathrm{Re}(h_{ij}) = \langle x_i, x_j\rangle = \langle y_i, y_j\rangle$. Since $z^i \wedge \bar{z}^j = x^i \wedge x^j - i(x^i \wedge y^j + x^j \wedge y^i) + y^i \wedge y^j$,

$$\omega = -\sum_{i<j} \mathrm{Im}(h_{ij})(x^i \wedge x^j + y^i \wedge y^j) + \sum_{i,j=1}^n \mathrm{Re}(h_{ij}) x^i \wedge y^j = \frac{i}{2} \sum_{i,j=1}^n h_{ij} z^i \wedge \bar{z}^j.$$

□

Definition. Let X be a complex manifold with Riemannian metric g. We say X admits a *Hermitian structure* if g is compatible with the induced almost complex structure I. A *Kähler structure* is an Hermitian structure for which the fundamental form $\omega = g(I(\cdot),\cdot)$ is closed, i.e., $d\omega = 0$.

Let us see some examples of Kähler manifolds. For a non-example, see page 10.

Example (Fubini-Study metric)**.** We describe a canonical Kähler structure on the projective space $\mathbf{P}^n$ called the *Fubini-Study metric.* Let $\{U_i, \kappa_i\}$ be the standard chart of $\mathbf{P}^n$, where U_i is the complement of $\{z = [z_1 : \cdots : z_n] \in \mathbf{P}^n : z_i = 0\}$. We define

$$\omega_i := \frac{i}{2\pi} \partial\bar{\partial} \log \left(\sum_{k=0}^n \left| \frac{z_k}{z_i} \right|^2 \right) \in \Omega^{1,1}(U_i).$$

To see that ω_i's glue to a globally defined form $\omega \in \Omega^{1,1}(X)$, we note that for any $z \in \mathbf{P}^n$ we have

$$\partial\bar{\partial} \log |z|^2 = \partial \left(\frac{1}{z\bar{z}} \bar{\partial}(z\bar{z}) \right) = \partial \left(\frac{z d\bar{z}}{z\bar{z}} \right) = \partial \left(\frac{d\bar{z}}{\bar{z}} \right) = 0.$$

Now $\omega_i|_{U_i \cap U_j} = \omega_j|_{U_i \cap U_j}$ follows from

$$\log \left(\sum_{k=0}^n \left| \frac{z_k}{z_j} \right|^2 \right) = \log \left(\left| \frac{z_i}{z_j} \right|^2 \sum_{k=0}^n \left| \frac{z_k}{z_i} \right|^2 \right) = \log \left(\left| \frac{z_i}{z_j} \right|^2 \right) + \log \left(\sum_{k=0}^n \left| \frac{z_k}{z_i} \right|^2 \right) = \log \left(\sum_{k=0}^n \left| \frac{z_k}{z_i} \right|^2 \right).$$

The global form ω is real, since $\overline{\partial\bar{\partial}} = \bar{\partial}\partial = -\partial\bar{\partial}$ yields $\omega_i = -\bar{\omega}_i$. It is also closed, since $\partial\omega_i = 0$. Following Proposition 3, since

$$\partial\bar{\partial} \log \left(1 + \sum_{k=1}^n |w_k|^2 \right) = \partial \left(\frac{\sum_k w_k d\bar{w}_k}{1 + \sum_k |w_k|^2} \right) = \frac{\sum_k dw_k \wedge d\bar{w}_k}{1 + \sum_k |w_k|^2} - \frac{(\sum_k \omega_k d\bar{w}_k)(\sum_l \bar{w}_l dw_l)}{(1 + \sum_k |w_k|^2)^2}$$

for $w_k = z_k/z_i = \kappa_i(z_k)$, we may write

$$\omega_i = \frac{i}{2\pi(1 + \sum_k |w_k|^2)^2} \sum_{k,l} h_{kl} dw_k \wedge d\bar{w}_l, \quad \text{where} \quad h_{kl} = \left(1 + \sum_k |w_k|^2 \right) \delta_{kl} - \bar{w}_i w_j.$$

where $w_k = z_k/z_i = \kappa_i(z_k)$. The Hermitian matrix h_{kl} is positive-definite. Indeed, for any $u \neq 0$,

$$u^T(h_{kl})\bar{u} = (u,u) + (w,w)(u,u) - u^T \bar{w} w \bar{u} = (u,u) + (w,w)(u,u) - |(w,u)|^2 > 0,$$

where $(\cdot,\cdot)$ is the standard Hermitian inner product on $\mathbf{C}^n$. Since I and ω determine the inner product g, we have produced a Riemannian metric g on $\mathbf{P}^n$ that is Kähler.

Note that if (X, g) is Kähler, then the restriction $g|_Y$ gives any complex submanifold $Y \subset X$ a Kähler structure. Therefore, any projective manifold is also Kähler. In particular, Riemann surfaces, which are projective due to a remarkable result of Riemann, are Kähler.

Before moving on, we state some crucial identities called *Kähler identities.* These identities express the commutator relations of various operators on a Kähler manifold. As we shall see later, in the context of Hodge theory, they reflect what is particularly nice about Kähler manifolds.

Definition. Let V be an n-dimensional oriented real vector space with $(I, \langle \cdot, \cdot \rangle, \omega)$. Let vol be the volume form. We define the following linear operators:

(i) *Lefschetz operator* $L : \wedge^k V^\vee \to \wedge^{k+2} V^\vee$ given by $\alpha \mapsto \omega \wedge \alpha$;

(ii) *Hodge* $*$*-operator* $* : \wedge^k V \to \wedge^{n-k} V$ given by $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \cdot \mathrm{vol}$;

(iii) *dual Lefschetz operator* $\Lambda : \wedge^k V^\vee \to \wedge^{k-2} V^\vee$ given by $\Lambda = *^{-1} \circ L \circ *$.

An element $\alpha \in \wedge^k V^\vee$ is *primitive* if $\Lambda \alpha = 0$. Write P^k for the subspace of primitive elements.

It is mostly straightforward (except for (iv), which requires some Lie theory) to verify the following:

(i) $*$ is self adjoint, i.e., $\langle \alpha, *\beta \rangle = (-1)^{k(n-k)} \langle *\alpha, \beta \rangle$;

(ii) L and Λ are adjoint, i.e., $\langle \Lambda \alpha, \beta \rangle = \langle \alpha, L\beta \rangle$;

(iii) L^k induces an isomorphism $\wedge^k V^\vee \to \wedge^{2n-k} V^\vee$;

(iv) there is the so-called *Lefschetz decomposition* $\wedge^k V^* = \oplus_{i \geq 0} L^i (P^{k-2i})$.

Furthermore, these operators can be extended $\mathbf{C}$-linearly when $(V, \langle \cdot, \cdot \rangle)$ becomes $(V_\mathbf{C}, \langle \cdot, \cdot \rangle_\mathbf{C})$, where L and Λ are of type $(1,1)$ and $(-1,-1)$, respectively, and $* : \wedge^{p,q} V \to \wedge^{n-p,n-q}$ is given by $\alpha \wedge *\bar{\beta} = \langle \alpha, \beta \rangle_\mathbf{C} \cdot \mathrm{vol}$. The Lefschetz decomposition respects the bidegree decomposition so that we have $P^k_\mathbf{C} = \oplus_{p+q=k} P^{p,q}$ where $P^{p,q} = P^k_\mathbf{C} \cap \wedge^{p,q} V^\vee$.

Now we pass from linear operators to differential operators on manifolds.

Definition. Let (X, g) be an oriented n-dimensional Riemannian manifold. We define

(i) *adjoint operator* $d^* : \Omega^k(X) \to \Omega^{k-1}(X)$ given by $d^* = (-1)^{n(k+1)+1} * \circ d \circ *$;

(ii) *Laplacian* $\Delta : \Omega^k(X) \to \Omega^k(X)$ given by $\Delta = d^* d + d d^*$.

A form α is *harmonic* if $\Delta \alpha = 0$. The space of all harmonic k-forms is denoted $\mathcal{H}^k(X)$.

If (X, g) admits a Hermitian structure, then n is even, so that d^* becomes $- * \circ d \circ *$. We define $\partial^* := - * \circ \bar{\partial} \circ *$ and $\bar{\partial}^* := - * \circ \partial \circ *$ analogously, which are of type $(-1, 0)$ and $(0, -1)$ respectively. Finally, we define *Laplacians* $\Delta_\partial := \partial^* \partial + \partial \partial^*$ and $\Delta_{\bar{\partial}} := \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$.

Now suppose X is compact. We define an inner product $\langle \cdot, \cdot \rangle$ on $\Omega^\bullet(X)$ by $\langle \alpha, \beta \rangle := \int_X \alpha \wedge *\beta$, with respect to which d and d^* are formally adjoint. Indeed, using Stokes' theorem we have

$$\langle d\alpha, \beta \rangle = \int_X d\alpha \wedge *\beta = (-1)^{k-1} \int_X \alpha \wedge d(*\beta) = (-1)^{(p-1)+n(n-p)} \int_X \alpha \wedge *(*d * \beta) = \langle \alpha, d^* \beta \rangle.$$

Similarly one can show that Δ is self-adjoint.

Example. Let us do some local computation to make everything concrete. Let $\alpha = \sum_{i=1}^n f_i dx_i$ be a locally defined smooth 1-form. Then

$$d^*\alpha = (-1)^{n(1+1)+1} * d * \alpha = (-1) * d \sum_{i=1}^n (-1)^{i-1} f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$
$$= (-1) * \sum_{i=1}^n \partial f_i/\partial x_i dx_1 \wedge \cdots \wedge dx_n = -\sum_{i=1}^n \partial f_i/\partial x_i.$$

As an exercise, show that $\Delta(f) = -\sum_{i=1}^n \partial^2 f/\partial x_i^2$ for a locally defined smooth 0-form, so that Δ corresponds to the usual notion of second-order differential operator.

The above discussion carries over to compact Hermitian manifolds, where we define on $\Omega^\bullet_{\mathbf{C}}(X)$ an inner product $(\cdot,\cdot) := \int_X \alpha \wedge *\bar{\beta}$, with respect to which ∂ and ∂^* are adjoint, and so do $\bar{\partial}$ and $\bar{\partial}^*$. Note also that as a consequence of Proposition-Definition 1.1, $d^* = \partial^* + \bar{\partial}^*$ and $(\partial^*)^2 = (\bar{\partial}^*)^2 = 0$.

Proposition 4 (Kähler identities)**.** *Let (X, g) be a Kähler manifold. Then the following hold:*

(i) $[\bar{\partial}, L] = [\partial, L] = 0$ *and* $[\bar{\partial}, \Lambda] = [\partial^*, \Lambda] = 0$*;*

(ii) $[\bar{\partial}^*, L] = i\partial$, $[\partial^*, L] = -i\bar{\partial}$ *and* $[\Lambda, \bar{\partial}] = -i\partial^*$, $[\Lambda, \partial] = i\bar{\partial}^*$*;*

(iii) $\Delta_\partial = \Delta_{\bar{\partial}} = \Delta/2$, *and* Δ *commutes with* $*$, ∂, $\bar{\partial}$, ∂^*, $\bar{\partial}^*$, L, *and* Λ.

The proof of these identities involve rather messy manipulations of symbols. See [2] for detail.

2 Hodge theory

2.1 Hodge theory for compact oriented Riemannian manifolds

A key motivation for Hodge theory is to find harmonic representatives of de Rham cohomology classes. In the sense that will be explained below, harmonic representatives are the ones with minimal norm with respect to the inner product $\langle \cdot,\cdot \rangle$ on $\Omega^\bullet(M)$. In the following, we provide a rough sketch of the idea, without worrying about concerns from analysis.

Let (M, g) be a compact oriented Riemannian manifold. For $\alpha \in \Omega^k(M)$, the de Rham cohomology class $[\alpha] \in H^k_{\text{dR}}(M)$ consists of forms $\alpha + d\beta$ for $\beta \in \Omega^{k-1}(M)$. Assuming everything is nice, we'd have an element $\alpha_0 \in [\alpha]$ of minimal norm with respect to $\langle \cdot,\cdot \rangle$, which is perpendicular to the $d\beta$, i.e., $\langle \alpha_0, d\beta \rangle = \langle d^*\alpha_0, \beta \rangle = 0$ for all $\beta \in \Omega^{k-1}(M)$. In other words, α_0 is a solution to the differential equations $d\alpha = 0$ and $d^*\alpha = 0$. Moreover, since d and d^* are adjoint, we have $\operatorname{Im} d^* \perp \ker f$ and $\operatorname{Im} d \perp \ker d^*$. These facts suggest an orthogonal decomposition of the form

$$\Omega^k(M) = (\ker d \cap \ker d^*) \oplus d(\Omega^{k-1}(M)) \oplus d^*(\Omega^{k+1}(M)),$$

where the first two terms represent closed forms. Therefore, every de Rham cohomology class is represented by a unique element in $\ker d \cap \ker d^*$. We claim that $\ker d \cap \ker d^* = \ker \Delta$ is the subspace $\mathcal{H}^k(M)$ of harmonic forms. Indeed, since $d : \Omega^k(M) \to d(\Omega^k(M))$ is surjective, the adjoint operator $d^* : d(\Omega^k(M)) \to \Omega^k(M)$ is injective. Similarly, since d^* is surjective on $\Omega^k(M)$, d is injective on $d^*(\Omega^k(M))$. Thus, the composition

$$d(\Omega^k(M)) \xrightarrow{d^*} \Omega^k(M) \xrightarrow{d} d(\Omega^k(M))$$

is injective. On the other hand, dd^* sends both $\ker d \cap \ker d^*$ and $d^*(\Omega^k(M))$ to 0. An analogous discussion shows that d^*d is injective on $d^*(\Omega^k(M))$ and sends everything else to 0. Therefore, the kernel of $\Delta = dd^* + d^*d$ is precisely $\ker d \cap \ker d^*$. By the previous discussion, every de Rham class thus has a harmonic representative whose norm is minimized.

Theorem 5 (Hodge decomposition for compact oriented Riemannian manifolds)**.** *Let (M, g) be a compact oriented Riemannian manifold. Then for every k we have:*

(i) $\mathcal{H}^k(M)$ is finite-dimensional;

(ii) there is an orthogonal decomposition $\Omega^k(M) = \mathcal{H}^k(M) \oplus \Delta(\Omega^k(M))$ with respect to $\langle \cdot, \cdot \rangle$.

Corollary 6. *There is an isomorphism $\mathcal{H}^k(M) \xrightarrow{\simeq} H^k_{\mathrm{dR}}(M)$. Equivalently, every de Rham class has a unique harmonic representative.*

It may seem that we have already proved Theorem 5 by our previous discussion, but we have in fact ignored some analytical problems. For starters, direct sum decomposition for an infinite-dimensional space is not entirely trivial. Moreover, the problem at the core is that the existence of an element of minimal norm is only true in generality for closed convex subspaces of a Hilbert space. So we have to complete $\Omega^k(M)$ to a Hilbert space with respect to the L^2-norm. Now the previous argument applies and yields a decomposition for L^2-forms. The essential question is then: how do we go back to smooth forms? This turns out to be a serious analytical challenge.

Luckily, a result from functional analysis states that a weak solution to a Laplace equation is in fact a true solution. To make it precise, fix any $\alpha \in \Omega^k(M)$ and consider the equation $\Delta\eta = \alpha$. A solution η_0 to this Laplace equation defines a linear functional $\ell : \Omega^k(M) \to \mathbf{R}$ given by $\ell(\beta) = \langle \alpha, \beta \rangle$ that satisfies $\ell(\Delta\beta) = \langle \eta_0, \Delta\beta \rangle = \langle \Delta\eta_0, \beta \rangle = \langle \alpha, \beta \rangle$ for any $\beta \in \Omega^k(M)$ by the self-adjointness of Δ. We call such an ℓ a *weak solution* of $\Delta\eta = \alpha$.

Lemma 7 (regularity lemma)**.** *If ℓ is a weak solution of $\Delta\eta = \alpha$ for some $\alpha \in \Omega^k(M)$, then there exists $\eta_0 \in \Omega^k(M)$ such that $\ell(\beta) = \langle \eta_0, \beta \rangle$ for any $\beta \in \Omega^k(M)$. In particular, $\Delta\eta_0 = \alpha$.*

Lemma 8. *Let $\{\alpha_n\}$ be a sequence of smooth p-forms on M such that $\|\alpha_n\| \leq C$ and $\|\Delta\alpha_n\| \leq C$ for all n and some constant $C > 0$. Then $\{\alpha_n\}$ has a subsequence that is Cauchy in $\Omega^k(M)$.*

Assuming these technical lemmas, we are able to sketch the proof of Theorem 5.

Proof sketch of Theorem 5. If $\mathcal{H}^k(M)$ were not finite-dimensional, then it would contain an infinite orthonormal sequence, which is impossible by Lemma 8. This proves (i). Now let $\{e_i\}_{i=1}^n$ be an orthonormal basis for $\mathcal{H}^k(M)$. For any $\alpha \in \Omega^k(M)$ we can write $\alpha = \gamma + \sum_{i=1}^n \langle \alpha, e_i \rangle e_i$ for some $\gamma \in (\mathcal{H}^k(M))^\perp$. It suffices to show that $(\mathcal{H}^k(M))^\perp = \Delta(\Omega^k(M))$. Since $\langle \Delta\alpha, \beta \rangle = \langle \alpha, \Delta\beta \rangle$ for any $\alpha \in \Omega^k(M)$ and $\beta \in \mathcal{H}^k(M)$, we have $(\mathcal{H}^k(M))^\perp \supset \Delta(\Omega^k(M))$.

For the reverse inclusion, consider $\alpha \in (\mathcal{H}^k(M))^\perp$. Define a linear function $\ell : \Delta(\Omega^k(M)) \to \mathbf{R}$ by $\ell(\Delta\beta) = \langle \alpha, \beta \rangle$ for $\beta \in \Omega^k(M)$. If ℓ is bounded, then by the Hahn-Banach ℓ can be extended to the whole of $\Omega^k(M)$. We have thus found a weak solution to the equation $\Delta\eta = \alpha$. By Lemma 7 there exists a true solution $\eta_0 \in \Omega^k(M)$, which implies that $\alpha \in \Delta(\Omega^k(M))$.

Hence, it suffices to prove that ℓ is a bounded linear functional. We claim that there is a constant $C > 0$ for which $\|\alpha\| \leq C\|\Delta\alpha\|$ for all $\alpha \in (\mathcal{H}^k(M))^\perp$. Suppose for contradiction that there exists a sequence $\{\alpha_i\}$ in $(\mathcal{H}^k(M))^\perp$ with $\|\alpha_i\| = 1$ and $\|\Delta\alpha_i\| \to 0$. By Lemma 8 we can assume that $\{\alpha_i\}$ is itself Cauchy, so that the linear functional $\kappa : \Omega^k(M) \to \mathbf{R}$ given by $\kappa(\beta) = \lim_{i\to\infty}\langle \alpha_i, \beta \rangle$ for $\beta \in \Omega^k(M)$ is well-defined. Moreover, κ is a bounded and is a weak solution to $\Delta\alpha = 0$. Indeed,

$\kappa(\Delta\beta) = \lim_{i\to\infty}\langle\alpha_i, \Delta\beta\rangle = \lim_{i\to\infty}\langle\Delta\alpha_i, \beta\rangle = 0$. By Lemma 7 there exists $\alpha \in \Omega^k(M)$ such that $\kappa(\beta) = \langle\alpha, \beta\rangle$. Therefore, $\alpha_i \to \alpha \in (\mathcal{H}^k(M))^\perp$. But this contradicts with $\Delta\alpha = 0$. Now we can use $C > 0$ to bound ℓ by $|\ell(\Delta\beta)| = |\langle\alpha, \beta\rangle| \le \|\alpha\|\,\|\beta\| \le C\|\Delta\alpha\|\,\|\beta\|$. Since $\alpha \in (\mathcal{H}^k(M))^\perp$ is fixed, we see that ℓ is indeed bounded. □

2.2 Hodge theory for compact Kähler manifolds

Of course, one has analogous results for compact Hermitian manifolds. Since the proof essentially uses the same techniques, we shall omit it and refer to [3].

Theorem 9. *Let X be a compact Hermitian manifold. There exists orthogonal decompositions*

$$\Omega^{p,q}(X) = \mathcal{H}^{p,q}_{\partial}(X) \oplus \Delta_\partial(\Omega^{p,q}(X)) \quad \text{and} \quad \Omega^{p,q}(X) = \mathcal{H}^{p,q}_{\bar\partial}(X) \oplus \Delta_{\bar\partial}(\Omega^{p,q}(X))$$

with respect to $(\cdot,\cdot)$ for any p, q. Moreover, $\mathcal{H}^{p,q}(X)$ is finite-dimensional.

Corollary 10. *The canonical projection $H^{p,q}_{\bar\partial}(X) \to H^{p,q}(X)$ is an isomorphism.*

For compact Kähler manifolds, Kähler identities show that $\mathcal{H}^{p,q}_{\partial}(X) = \mathcal{H}^{p,q}_{\bar\partial}(X)$.

Corollary 11 (Hodge decomposition)**.** *Let (X, g) be a compact Kähler manifold. Then there is a decomposition $H^k(X, \mathbf{C}) = \oplus_{p+q=k} H^{p,q}(X)$ that does not depend on the Kähler structure.*

Proof. Using Corollary 10, we have $H^k(X,\mathbf{C}) = \mathcal{H}^k(X)_{\mathbf{C}} = \oplus_{p+q=k}\mathcal{H}^{p,q}(X) = \oplus_{p+q=k}H^{p,q}(X)$ which *a priori* might depend on the choice of the Kähler metric g. Let g' be another Kähler metric and write $\mathcal{H}^{p,q}(X,g)$ and $\mathcal{H}^{p,q}(X,g')$ for harmonic forms with respect to different metrics. Let $\alpha \in \mathcal{H}^{p,q}(X,g)$ and $\alpha' = \alpha + \bar\partial\beta \in \mathcal{H}^{p,q}(X,g')$ be such that they represent the same class in $H^{p,q}(X)$. We wish to show that they also represent the same class in $H^k(X,\mathbf{C})$. Indeed, since $d\bar\partial\beta = d(\alpha' - \alpha) = 0$ and that $\bar\partial\beta$ is orthogonal to $\mathcal{H}^k(X,g)_{\mathbf{C}}$, by the Hodge theory for Riemannian manifolds we have $\bar\partial\beta \in d(\Omega^{k-1}_{\mathbf{C}}(X))$. Therefore, $[\alpha] = [\alpha']$ in $H^k(X,\mathbf{C})$. □

Recall that there are operators L, Λ, and $*$ on the level of forms that induce decompositions and isomorphisms. For a compact Kähler manifold X of dimension n, these decompositions and isomorphisms are compatible with the Hodge decomposition. For instance, using $[\Delta, L] = 0$ from the Kähler identities, we see that L maps harmonic forms to harmonic forms. Since L^{n-k} is a bijection on $\Omega^k(X)$, the induced map $L^{n-k} : \mathcal{H}^{p,k-p}(X) \to \mathcal{H}^{n+p-k,n-p}(X)$ is injective. Surjectivity follows from considering the dual operator Λ. Note this won't hold hold if the underlying manifold is not Kähler. Note also that the operators $L : H^{p,q}(X) \to H^{p+1,q+1}(X)$ and $\Lambda : H^{p,q}(X) \to H^{p-1,q-1}(X)$ only depend on the Kähler class $[\omega]$.

Definition. The *primitive cohomology* $H^{p,q}(X)_{\text{prim}}$ is the kernel of $\Lambda : H^{p,q}(X) \to H^{p-1,q-1}(X)$.

Theorem 12 (Hard Lefschetz theorem)**.** *There is a decomposition*

$$H^k(X,\mathbf{R}) = \oplus_{i\ge 0} L^i H^{k-2i}(X,\mathbf{R})_{\text{prim}}$$

that respects the Hodge decomposition, i.e., $H^k(X,\mathbf{R})_{\text{prim}} \otimes \mathbf{C} = \oplus_{p+q-k} H^{p,q}(X)_{\text{prim}}$.

Similarly, the Hodge $*$-operator which depends on $[\omega]$ acts naturally on $H^\bullet(X,\mathbf{C})$ and induces isomorphisms $* : H^{p,q}(X) \to H^{n-q,n-p}(X)$. This motivates us to consider the pairing

$$(\alpha,\beta) \mapsto \int_X \alpha \wedge \beta : \mathcal{H}^{p,q}_{\bar\partial}(X) \times \mathcal{H}^{n-p,n-q}_{\bar\partial}(X) \to \mathbf{C}.$$

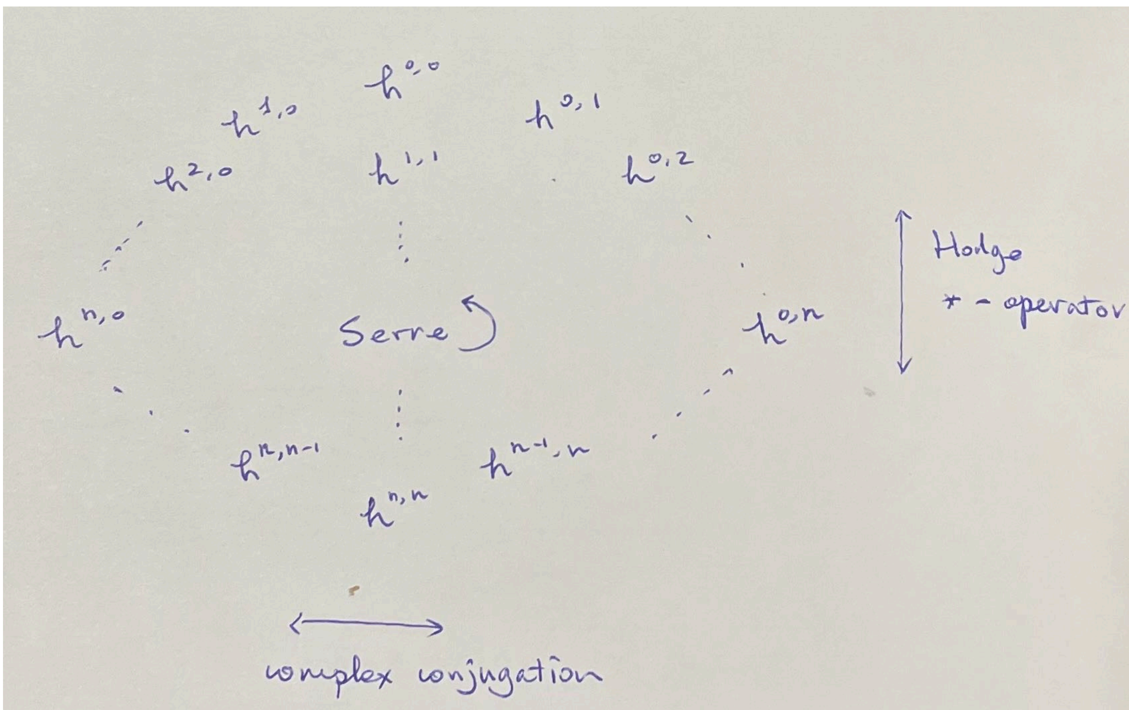
Since for $\alpha \ne 0$, we have $*\bar\alpha \in \mathcal{H}^{n-p,n-q}_{\bar\partial}(X)$ and $\int_X \alpha \wedge *\bar\alpha = \|\alpha\|^2 > 0$, this pairing is non-degenerate, so that we have *Serre duality* $\mathcal{H}^{p,q}_{\bar\partial}(X) \cong \mathcal{H}^{n-p,n-q}_{\bar\partial}(X)^\vee$ on the level of harmonic forms.

Remark. This is a special case of what is often referred to as Serre duality. In fact, our discussion of Hodge theory can be generalized for a holomorphic vector bundle E with a Hermitian structure h on a compact Hermitian manifold X. Accordingly, we define differential operators and Hermitian inner product. The present discussion is then recovered by taking E to be the trivial bundle with constant Hermitian structure. In the twisted version, there is a similar non-degenerate natural pairing $H^{p,q}(X,E) \times H^{n-p,n-q}(X,E^\vee) \to \mathbf{C}$. Taking $p = 0$ and applying the so-called *Dolbeault's theorem* $H^{p,q}(X) \cong H^q(X, E \otimes \Omega_X^p)$, we recover the common formulation of Serre duality

$$H^q(X,E) \cong H^{0,q}(X,E) \cong H^{n,n-q}(X,E^\vee)^\vee \cong H^{n-q}(X,E^\vee \otimes K_X)^\vee,$$

where K_X is the canonical bundle of X. Note that generalizing everything to the twisted version involves very little additional work.

Let $h^{p,q} = \dim H^{p,q}(X)$. We can summarize our previous discussion with the following diagram:



Complex conjugation $\overline{H^{p,q}}(X) = H^{q,p}(X)$ and Hodge $*$-operator provides symmetry under horizontal and vertical reflections respectively. Serre duality guarantees that the diagram is invariant under rotation by π. Due to its unique diamond-like shape and symmetry, this diagram is often referred to as the *Hodge diamond*.

Remark. From a somewhat high-tech viewpoint, Hodge decomposition can be seen as the degeneration of a spectral sequence. There is a spectral sequence called the *Frölicher spectral sequence* (or *Hodge-to-de Rham spectral sequence*) of the form

$$E_1^{p,1} := H^{p,q}(X) \Rightarrow H^{p+q}_{\mathrm{dR}}(X)$$

with $E_\infty^{p,q} := \mathrm{Gr}^p H^{p+q}(X)$ the associated graded piece of the so-called Hodge filtration defined by $F^p H^k(X,\mathbf{C}) = \mathrm{Im}(H^{k,\bullet+p}(X) \to H^{k,\bullet}(X))$. From this perspective, Hodge decomposition of a compact Kähler manifold is simply the degeneration of this spectral sequence at the E_1 page.

2.3 Applications

There are numerous applications of Hodge theory, like Serre duality that we have discussed. Note that with Hodge theory, we also get Poincaré duality for Riemannian manifolds for free. Indeed, since $[\Delta, *] = 0$, for any harmonic representative α of a nonzero cohomology class in $H^k(X)$, we have a similar non-degenerate pairing

$$(\alpha, *\alpha) \mapsto \int_X \alpha \wedge *\alpha : H^k(X) \times H^{n-k}(X) \to \mathbf{R}.$$

The reason for this is that Hodge theory is a *refinement* of the de Rham theory. We are given a canonical choice of cohomology class, the harmonic representative, in each de Rham group, and with this extra piece of data we can do much more.

Let $b_k := \dim_{\mathbf{C}} H^k(X, \mathbf{C})$ be the Betti numbers. Another quick application of Hodge theory allows us to disprove that certain manifolds are Kähler.

Proposition 13. *The odd Betti numbers b_{2k+1} of a compact Kähler manifold X are even.*

Proof. This follows from the fact that $\overline{H^{p,q}}(X) = H^{q,p}(X)$ and $b_{2k+1} = \sum_{p+q=2k+1} h^{p,q}$. □

Example. Recall the Hopf manifold defined on page 2. Consider the two-dimensional case, which is a compact quotient of $\mathbf{C}^2 \setminus \{0\}$ by the free action of $\mathbf{Z}$ via $(z_1, z_2) \mapsto (\lambda z_1, \lambda z_2)$. Since $\mathbf{C}^2 \setminus \{0\}$ is simply connected, $\pi(X) = \mathbf{Z}$. By the Hurewicz theorem $H^1(X, \mathbf{Z}) = \mathbf{Z}$, and thus $b_1(X) = \mathbf{Z}$. Therefore, Hopf surfaces are not Kähler. In particular, they are not projective.

The strength of Hodge structure as an invariant becomes evident in the Torelli theorem for compact Riemann surfaces. To state it, we formalize the notion of Hodge structure.

Definition. A *rational Hodge structure of weight k* consists of a rational vector space H and a decomposition $H \otimes_{\mathbf{Q}} \mathbf{C} = \oplus_{p+q=k} H^{p,q}$ satisfying $\overline{H}^{p,q} = H^{q,p}$. A *polarization* of H is a bilinear form $(\cdot, \cdot) : H \times H \to \mathbf{Q}$ such that

(i) $(z^k \alpha, z^k \beta) = (z\bar{z})^k (\alpha, \beta)$ for $z \in \mathbf{C}^\times$, and

(ii) $(-i)^k (\cdot, \cdot)$ is symmetric and positive-definite.

Note that the data of a rational Hodge structure is equivalent to the data of a real representation $\rho : \mathbf{C}^\times \to \mathrm{GL}(H_{\mathbf{R}})$. The motivation for such a definition comes from Hodge decomposition and the following relation.

Proposition 14 (Hodge-Riemann bilinear relation)**.** *Let X be an n-dimensional compact Kähler manifold with Kähler class $[\omega]$ and nonzero $\alpha \in H^{p,q}(X)_{\mathrm{prim}}$. Then*

$$i^{p-q}(-1)^{(p+q)(p+q-1)/2} \int_X \alpha \wedge \bar{\alpha} \wedge [\omega]^{n-p-q} > 0.$$

Recall that we have an exponential short exact sequence on a complex manifold X of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\mathbf{Z}} & \hookrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X^\times & \longrightarrow & 0 \\ & & & & f & \longmapsto & \exp(2\pi i f) & & \end{array}$$

where $\underline{\mathbf{Z}}$ is the locally constant sheaf and $\mathcal{O}_X$ the sheaf of holomorphic functions on X. The Picard group $\mathrm{Pic}(X)$ is identified with $H^1(X, \mathcal{O}_X^\times)$, and we define the *Jacobian* $J(X)$ of to be the kernel of $\mathrm{Pic}(X) \to H^2(X, \mathbf{Z})$ in the associated long exact sequence, i.e., $J(X) = H^1(X, \mathcal{O}_X)/H^1(X, \mathbf{Z})$. If X is compact Kähler, then using Hodge decomposition one can show that $J(X)$ is in a natural way a complex torus of dimension $b_1(X)$.

Theorem 15 (Torelli's theorem)**.** *A compact Riemann surface is determined by its Jacobian.*

Assuming this result, we shall prove the following.

Corollary 16. *A compact Riemann surface is determined by its polarized Hodge structure.*

Proof. For a compact Riemann surface C, a polarized Hodge structure is the datum of the lattice $H^0(C, \mathbf{Z})$ in $H^0(C, \Omega^1_C)^\vee$ with a cup product form $(\cdot, \cdot)$ on $H^1(C, \mathbf{Z})$ satisfying the Hodge-Riemann bilinear relations $(\alpha, \beta) = \int_C \alpha \wedge \beta = 0$ and $i(\alpha, \bar{\alpha}) > 0$ for any $\alpha, \beta \in \Omega^{1,0}(C)$. Given a polarized Hodge structure then determines the Jacobian variety $J(C) = H^1(C, \mathcal{O}_C)/H^1(C, \mathbf{Z})$, where we have used the identification $H^0(C, \Omega^1_C)^\vee \cong H^1(C, \mathcal{O}_C)$ from Serre duality. □

The upshot is that Hodge theory encapsulates interesting linear algebraic structure that can be used to study geometric problems. It can also function as a useful invariant.

We end our discussion with a famous conjecture. Let X be a compact projective n-manifold.

Definition. The *fundamental class* $[Z] \in H^{p,p}(X)$ of a complex submanifold $Z \subset X$ of codimension p in X is defined by the condition

$$\int_X \alpha \wedge [Z] = \int_X \alpha|_Z$$

for all $\alpha \in H^{2n-2p}(X)$. A class in $H^{p,p}(X, \mathbf{Q}) := H^{p,p}(X) \cap H^\bullet(X, \mathbf{Q})$ is called *analytic* if it is contained in the $\mathbf{Q}$-vector space generated by all fundamental classes $[Z] \in H^{p,p}(X, \mathbf{Z})$.

Hodge Conjecture. Any class in $H^{p,p}(X, \mathbf{Q})$ is analytic.

This is recognized as one of the seven Millennium Problems. If you happen to prove it, don't forget to claim your prize of 1 million dollars from the Clay Mathematics Institute. The following theorem provides evidence for Hodge Conjecture to be true.

Theorem 17 (Lefschetz theorem on $(1,1)$-classes)**.** *Let X be a compact Kähler manifold. Then* $\text{Pic}(X) \to H^{1,1}(X, \mathbf{Z})$ *is surjective*

Proof. Identifying $\text{Pic}(X)$ with $H^1(X, \mathcal{O}_X^\times)$, it suffices to show that $i : H^2(X, \mathbf{Z}) \to H^2(X, \mathcal{O}_X)$ induced by the exponential short exact sequence is the zero map. By Hodge theory we have $H^2(X, \mathcal{O}_X) \cong H^{0,2}(X)$. Observe that i factors through $H^2(X, \mathbf{C})$, and on which i restricts to the projection onto $H^{0,2}(X)$. Therefore i is zero on $H^2(X, \mathbf{Z}) \cap H^{1,1}(X)$. □

It can be shown that the image of a line bundle $\mathcal{O}(D)$ associated to a Weil divisor D under the surjection $\text{Pic}(X) \to H^{1,1}(X, \mathbf{Z})$ is $[D]$. Using the Kodaira vanishing theorem, we see that any line bundle on a projective manifold is associated to a divisor. Therefore, any class in $H^{1,1}(X, \mathbf{Z})$ can be written as a linear combination of fundamental classes of hypersurfaces. This combined with the Hard Lefschetz theorem shows that the Hodge Conjecture is true for X of dimension ≤ 3.

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