# The structure theory of complete local rings

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## 1 Introduction.

To quote Hochster [Hoch]: in the study of of commutative Noetherian rings, localization at a prime followed by completion at the resulting maximal ideal is a way of life. This naturally leads to the notion of a complete local ring. We shall present a general structural result of those rings, commonly called the Cohen structure theorem, which gives a very nice characterization of complete local rings that are ubiquitous in the study of commutative algebra and algebraic geometry.

Roughly speaking, the theorem states that any complete local Noetherian ring $R$ is a homomorphic image of a formal power series ring in finitely many variables over a "nice" ring. If $R$ contains a field (in which case $R$ is said to be *equicharacteristic*), then this ring can be taken to be the residue class field $K$. We shall only prove the theorem in the equicharacteristic case. But we will briefly state the theorem in the general setting at the end. We will also mention a few consequences of the Cohen structure theorem, including Kunz's theorem.

Before diving into the details, I plan to talk briefly outline the proof goes. First we need an important definition.

**Definition.** Let $(R, \mathfrak{m}, K)$ be an equicharacteristic local ring. A subfield $K_0 \subset R$ is called a *coefficient field* if the natural map $K_0 \hookrightarrow R \to R/\mathfrak{m}$ is an isomorphism.

To prove the Cohen structure theorem, one needs to establish the existence of a coefficient field for any complete local Noetherian ring. There are three different cases to consider where the proof is different for each. If the complete local ring contains a field of characteristic zero, then the coefficient field is actually the maximal element of the set of all subfields of $R$. An important step in the argument is that complete local rings satisfy Hensel's lemma, which we shall prove.

If the complete local ring $R$ in question is of prime characteristic $p > 0$, and that its residue class field $K = R/\mathfrak{m}$ is perfect, then we have a nice description of the coefficient fields explicitly:

$$K_0 = \bigcap_{n \geq 0} R^{p^n}$$

is the unique coefficient field of $R$. The proof here is completely different from the characteristic-0 case. The proof is analytic in nature, and the key idea is to construct a certain Cauchy sequence in the analytic topology.

Finally, if the residue class field $K$ is not perfect, then we need the notion of a *p-basis* of $K$ over $K^p$, which is a maximal subset of "independent" generators one needs to construct $K$ from $K^p$. In this case, we also have an explicit description of what the coefficient fields look like: let $T$ be a lift of the $p$-basis of $K/K^p$, then

$$K_0 = \bigcap_{n \geq 0} R^{p^n}[T]$$

is the unique coefficient field of $R$ that contains $T$. The proof is slightly more technical due to the complications of $p$-basis, but the idea is similar to the previous perfect case, which is expected because the descriptions of the coefficient field look analogous. A subtle difference that is immediate from the description is that while the coefficient field is unique if $K$ is perfect, it is only unique up to a choice of the lift of a $p$-basis if $K$ is imperfect.

With that being said, let us assume for now that the hard part of the proof, which is the existence of coefficient fields, and proceed from there. We shall see that the Cohen structure theorem follows without too much difficulty.

The general setup is is the following. Let $(A, \mathfrak{m}, K) \to (B, \mathfrak{n}, L)$ be a local homomorphism of complete local rings. Suppose that $f_1, \cdots, f_n \in \mathfrak{n}$ together with $\mathfrak{m}B$ generate an $\mathfrak{n}$-primary ideal $\mathfrak{q}$ in $B$. There is a natural map $A[X_1, \cdots, X_n] \to B$ obtained by sending $x_i$ to $f_i$.

**Proposition 1.1.** *There exists a unique continuous homomorphism $A[[X_1, \cdots, X_n]] \to B$ that extends the map $A[X_1, \cdots, X_n] \to B$.*

*Proof.* The proposition follows from the following general fact: if $\phi : R \to S$ is a ring homomorphism with $\phi(I) \subset J$, and if $S$ is $J$-adically complete and separated, then there exists a unique induced homomorphism $\hat{R}_I \to S$ that is continuous. Indeed, Cauchy sequences are mapped to Cauchy sequences. Note that the fact applies here because $B$ is $\mathfrak{n}$-adically separated. $\square$

**Proposition 1.2.** *Suppose that $L/K$ is a finite extension. Then $B$ is finite over the image of the map $A[[X_1, \cdots, X_n]] \to B$.*

*Proof.* Since $\mathfrak{q} = (f_1, \cdots, f_n, \mathfrak{m}B)$ is $\mathfrak{n}$-primary and $\mathfrak{n}$ is a maximal ideal, there exists some $N$ such that $\mathfrak{n}^N \subset \mathfrak{q}$. Thus $B/\mathfrak{q}$ is finite dimensional over $L$, and thus finite dimensional over $K$ since $L/K$ is finite. We claim that an $L$-basis for $B/\mathfrak{q}$ lifts to an $A[[X_1, \cdots, X_n]$-basis of $B$. Observe that $B$ is $\mathfrak{q}$-adically separated since it is $\mathfrak{n}$-adically separated and $\mathfrak{q} \subset \mathfrak{n}$, so we are in the situation of the following lemma, which concludes the proof. $\square$

**Lemma 1.3.** *Let $A$ be a ring and $I \subset A$ an ideal. Let $M$ be an $A$-module. Suppose that $A$ is $I$-adically complete and that $M$ is $I$-adically separated. If $M/IM$ is generated by $\bar{w}_1, \bar{w}_2, \cdots, \bar{w}_n$ over $A/I$, then $M$ is generated by $w_1, \cdots, w_n$ over $A$, where $w_i \in M$ is the inverse image of $\bar{w}_i$.*

*Proof.* Let us write $M = \sum_i Aw_i + IM$, so that

$$M = \sum_i Aw_i + I\left(\sum_i Aw_i + IM\right) = \sum_i Aw_i + I^2M.$$

Inductively, we have $M = \sum_i Aw_i + I^\nu M$ for any $\nu > 0$. For any $\xi \in M$, write $\xi = \sum_i a_i w_i + \xi_1$ where $\xi_1 \in IM$. Then, we can write $\xi_1 = \sum_i a_{1,i} w_i + \xi_2$, where $a_{1,i} \in I$ and $\xi_2 \in I^2 M$. Inductively, choosing $a_{\nu,i} \in I^\nu$ and $\xi_{\nu+1} \in I^{\nu+1}M$ to satisfy $\xi_\nu = \sum_i a_{\nu,i} w_i + \xi_{\nu+1}$ for all $\nu \geq 1$. Since $A$ is $I$-adically complete, the sum $a_i + a_{1,i} + a_{2,i} + \cdots$ converges to an element $b_i \in A$. Since $M$ is $I$-adically separated, we have

$$\xi - \sum_i b_i w_i \in \bigcap_{\nu \geq 0} I^\nu M = (0).$$

□

**Corollary 1.4.** *If $K \to L$ is an isomorphism and $\mathfrak{q} = \mathfrak{n}$, then the natural map $A[[X_1, \cdots, X_n]] \to B$ is surjective.*

*Proof.* By the previous proposition, $1 \in R$ generates $R$ as an $A[[X_1, \cdots, X_n]]$-module. □

Now, if we assume the existence of a coefficient field for an equicharacteristic complete local ring, then we are in the case of the previous corollary, which proves the following theorem.

**Corollary 1.5** (Cohen, equicharacteristic)**.** *Let $(R, \mathfrak{m}, K)$ be an equicharacteristic complete local ring with coefficient field $K_0 \cong K$. Let elements $f_1, \cdots, f_n \in \mathfrak{m}$ generate $\mathfrak{m}$. Then the natural map $K[[X_1, \cdots, X_n]] \to R$ sending $X_i$ to $f_i$ is surjective.*

**Remark.** If we assume the existence of a coefficient field, then the remaining of the proof is rather straightforward. The crux of it is really Lemma 1.3, which is a Nakayama-type of result about lifting generators. In general, you are expected to see multiple “lifting” type of results and statements throughout the proof of the Cohen structure theorem. The general strategy of proof is to leverage the nice things about the analytic topology of a complete local ring.

# 2 Existence of coefficient field.

In this section, we will establish the existence of a coefficient field for an equicharacteristic complete local ring. As mentioned in the introduction, We divide the problem into three cases.

## 2.1 Case 1. Characteristic 0.

We need a fact that complete local rings are Henselian. This is a result about lifting polynomial factorizations.

**Theorem 2.1** (Hensel's lemma)**.** *Let $(R, \mathfrak{m}, K)$ be a complete local ring. Let $F(X) \in R[X]$ be a monic polynomial and $\bar{F} \in K[X]$ be the polynomial obtained by reducing the coefficients of $F$ modulo $\mathfrak{m}$. If there are coprime monic polynomials $g, h \in K[X]$ such that $\bar{F} = gh$, then there exist monic polynomials $G, H \in R[X]$ such that $F = GH$, $\bar{G} = g$, and $\bar{H} = h$.*

*Proof.* The proof is done by induction. Let $G_1, H_1 \in R[X]$ such that $\bar{G}_1 = g$, $\bar{H}_1 = h$, and $F \equiv G_1 H_1 \mod \mathfrak{m}$. Suppose that monic polynomials $G_n, H_n \in R[X]$ have been constructed such that $F \equiv G_n H_n \mod \mathfrak{m}^n$, $\bar{G}_n = g$, and $\bar{H}_n = h$. Then one can write

$$F - G_n H_n = \sum_i \omega_i U_i(X)$$

where $\omega_i \in \mathfrak{m}^n$, and since the polynomials are all monic, we must have $\deg U_i < \deg F$. Now, since $g$ and $h$ are coprime, there exists $v_i, w_i \in K[X]$ such that $\bar{U}_i = gv_i + hw_i$. Replacing $v_i$ be its remainder modulo $h$ and making the corrections to $w_i$, we can assume that $\deg v_i < \deg h$. In that case, $\deg hw_i = \deg(\bar{U}_i - gv_i) < \deg F$, so that $\deg w_i < \deg g$. Now we can choose $V_i, W_i \in R[X]$ such that $\bar{V}_i = v_i$, $\deg V_i = \deg v_i$, $\bar{W}_i = w_i$, and $\deg W_i = \deg w_i$. If we set

$$G_{n+1} = G_n + \sum_i \omega_i W_i, \quad \text{and} \quad H_{n+1} = H_n + \sum_i \omega_i V_i,$$

then, $F \equiv G_{n+1} H_{n+1} \mod \mathfrak{m}^{n+1}$, $\bar{G}_{n+1} = g$, and $\bar{H}_{n+1} = h$. Since $R$ is $\mathfrak{m}$-adically complete, simply taking $G = \lim_n G_n$ and $H = \lim_n H_n$ suffices. □

**Proposition 2.2.** *Let $(R, \mathfrak{m}, K)$ be a complete local ring that contains a field of characteristic $0$. Then $R$ has a coefficient field.*

*Proof.* Consider the set of all subfields of $R$, which is nonempty by assumption. By Zorn's lemma, there exists a maximal element $K_0$. Since $\mathfrak{m}$ consists of non-units, the natural map $K_0 \hookrightarrow R \to R/\mathfrak{m}$ is injective. We will show that the map is also surjective, so that $K_0$ is indeed a coefficient field. Let $K_0'$ be the image of $K'$ in $K$ under this map and let $\theta \in K \setminus K_0'$.

Suppose that $\theta$ is transcendental over $K_0'$. Let $t \in R$ be a lift of $\theta$ in $R$. Any element in $f(t) \in K_0[t] \subset R$ is invertible, since if otherwise $f(t) \in \mathfrak{m}$, then we get an algebraic equation of dependence for $\theta$ in $R/\mathfrak{m}$. Thus we obtain a field $K_0(t) \subset R$ is is strictly larger than $K_0$, which is a contradiction.

Suppose that $\theta$ is algebraic over $K_0'$. Let $f$ be the minimal polynomial of $\theta$ over $K_0$. Since $K_0$ has characteristic 0, $\theta$ is separable over $K_0'$, so that $f$ factors into a product $(X-\theta)H(X)$ of coprime polynomials in $K_0[X]$. By Hensel's lemma, the factorization has a unique lift $(X-t)h(X)$ to $R[X]$ where $t \in R$ is the lift of $\theta$ in $R$. We claim that $K_0[t] \subset R$ is a field that strictly contains $K_0$, thereby contradicting the maximality of $K_0$. Indeed, it suffices to show that $K_0[t] \cong K_0'[\theta]$. The map $K_0[t] \hookrightarrow R \to R/\mathfrak{m}$ has image $K_0'[\theta]$, so we only need to show that it is injective. Indeed, if $P(X) \in K_0[X]$ is such that $P(t)$ is mapped to 0, then $f$ divides $P(X)$, which implies that $P(t) = 0$. □

**Remark.** In this case, the choice of a coefficient field, which is given by Zorn's lemma, is not unique. Indeed, there are infinitely many ways of embedding $k(t)$, which is the coefficient field of $R = k(t)[[t]]$, into $R$, since $t$ is transcendental over $k$.

## 2.2 Case 2. Characteristic $p > 0$, and the residue class field is perfect.

In contrast to the previous case, the coefficient field is unique, and we also have an explicit description of what it must be, which is quite nice.

**Proposition 2.3.** *Let $(R, \mathfrak{m}, K)$ be a complete local ring of positive prime characteristic $p$. Suppose that $K$ is perfect. Then*

$$K_0 = \bigcap_{n \geq 0} R^{p^n}$$

*is the unique coefficient field for $R$.*

*Proof.* We claim that $K_0 \cap \mathfrak{m} = \{0\}$. Indeed, if $u \in K_0 \cap \mathfrak{m}$, then $u$ is a $p^n$-th power for all $n$. If $u = v^{p^n}$, then $v \in \mathfrak{m}$, so $u \in \cap_{n \geq 0} \mathfrak{m}^{p^n}$. But $\cap_{n \geq 0} \mathfrak{m}^{p^n} = (0)$, because $\mathfrak{p}^p \subset \mathfrak{m}$. Thus, every nonzero element of $K_0$ is a unit of $R$. Moreover, if $u = v^{p^n} \in K_0$, then $1/u = (1/v)^{p^n} \in K_0$, so $K_0 \subset R$ is indeed a field.

Now that the natural map $K_0 \to K$ is injective, it remains to check that it is surjective. Let $\theta \in K$. Let $r_n \in R$ be the element that maps to $\theta^{1/p^n} \in K$. Then $r_n^{p^n}$ maps to $\theta$. We claim that $\{r_n^{p^n}\}_n$ is a Cauchy sequence in $R$. Indeed, since both $r_n$ and $r_{n+1}^p$ map to $\theta^{1/p^n}$ in $K$, we must have $r_n - r_{n+1}^p \in \mathfrak{m}$. Taking $p^n$ powers, we get

$$r_n^{p^n} - r_{n+1}^{p^{n+1}} \in \mathfrak{m}^{p^n}.$$

The sequence has a limit $r \in R$ since $R$ is complete, and $r$ maps to $\theta$. Finally, it suffices to show that $r \in R^{p^k}$ for every $k$. To do so, consider the sequence $r_k, r_{k+1}^p, r_{k+2}^{p^2}, \cdots$. Each terms of this sequence maps to $\theta^{1/p^k}$, so the same argument shows that the sequence is Cauchy. The limit of the sequence $s_k \in R$ satisfies $s_k^{p^k} = r$ by construction, so $r \in R^{p^k}$, as desired.

Finally, we need to show uniqueness. Suppose $L$ is any coefficient field of $R$. Since $L \cong K$, it is perfect, so $L = L^{p^n}$ for all $n$. But then $L \subset L^{p^n} \subset R^{p^n}$ for all $n$, so $L \subset K_0$. But $K_0 \subset K \cong L$, so $K_0 = L$. □

**Remark.** One might be tempted to ask: is there some conceptual explanation for why the coefficient fields are not unique in the characteristic-0 case, but suddenly becomes unique here? After all, fields of characteristic 0 are perfect too. The reason behind is that in the characteristic-$p$ case there is a built-in structure of a Frobenius, which makes the lifting theory rather rigid. In the characteristic-0 case, no such canonical lifting tool exists, and thus the lack of uniqueness in the coefficient field.

## 2.3 Case 3. Characteristic $p > 0$, and the residue class field is imperfect.

Contrary to the previous case, if $K$ is not perfect, then the choice of a coefficient field is *not* unique. The proof strategy of leveraging Cauchy sequences in this case is similar, but now it is slightly more technical and we have to introduce the notion of a $p$-basis.

Let $K$ be an imperfect field of characteristic $p > 0$. Elements $\theta_1, \cdots, \theta_n$ in $K$ are called *p-independent* over $K^p$ if

$$K^p \subset K^p[\theta_1] \subset K^p[\theta_1, \theta_2] \subset \cdots \subset K^p[\theta_1, \cdots, \theta_n]$$

is a strictly increasing tower of fields. Since $K$ is imperfect, this means that at each stage the field extension is purely inseparable of degree $p$. A maximal $p$-independent subset of $K$ over $K^p$ is called a *p-basis* for $K/K^p$. Note that the existence of a $p$-basis is guaranteed by Zorn's lemma. If $\Theta$ is a $p$-basis for $K/K^p$, then we make the following observations:

(i) $K = K^p[\Theta]$, and every element of $K$ is uniquely expressible as a polynomial in $\Theta$ with coefficients in $K^p$ of degree at most $p-1$;

(ii) $\Theta^p$ is a $p$-basis of $K^p$ over $K^{p^2}$;

(iii) if we consider the consecutive extension $K^{p^2} \subset K^p \subset K$, then every element of $K$ is uniquely expressible as a polynomial in $\Theta$ with coefficients in $K^{p^2}$ of degree at most

$$p(p-1) + (p-1) = p^2 - 1;$$

(iv) inductively, monomials in $\Theta$ of degree at most $p^n - 1$ are a basis for $K$ over $K^{p^n}$ for any $n$.

**Proposition 2.4.** *Let $(R, \mathfrak{m}, K)$ be a complete local ring of positive prime characteristic $p$, and let $\Theta$ be a $p$-basis for $K/K^p$. Let $T \subset R$ be a lift of the $p$-basis to $R$. Then*

$$K_0 = \bigcap_{n\geq 0} R^{p^n}[T]$$

*is the unique coefficient field for $R$ that contains $T$.*

*Proof.* We follow the similar strategy as the proof in the case when $K$ is perfect. We claim that $K_0 \cap \mathfrak{m} = \{0\}$. We first make the observation that every element of $R^{p^n}[T]$ can also be written as polynomials in $T$ with coefficients in $R^{p^n}$ of degree at most $p^n - 1$. Indeed, any $N \in \mathbf{Z}_{\geq 0}$ can be written as $ap^n + b$ with $b \leq p^n - 1$, so that $t^N$ can be written as $(t^a)^{p^n} t^b$, where we view $(t^a)^{p^n}$ as absorbed into the coefficient in $R^{p^n}$.

Now, let $u \in R^{p^n}[T] \cap \mathfrak{m}$, and represent it with a polynomial in $T$ of degree at most $p^n - 1$ with coefficients in $R^{p^n}$. Its image in $K$, which is a polynomial in $\Theta$ of degree at most $p^n - 1$ with coefficients in $K^{p^n}$, is 0. Therefore, each coefficient of the polynomial representing $u$ must be in $R^{p^n} \cap \mathfrak{m} \subset \mathfrak{m}^{p^n}$. Thus

$$K_0 \cap \mathfrak{m} = \bigcap_{n\geq 0} (R^{p^n}[T] \cap \mathfrak{m}) \subset \bigcap_{n\geq 0} \mathfrak{m}^{p^n} = (0).$$

It suffices to check that the natural map $K_0 \to K$ is surjective. Let $\lambda \in K$. Since $K = K^{p^n}[\Theta]$, for each $n$ we choose $r_n \in R^{p^n}[T]$ that maps to $\lambda$ modulo $\mathfrak{m}$. The sequence $\{r_n\}_n$ is Cauchy since $r_n - r_{n+1} \in R^{p^n} \cap \mathfrak{m} \subset \mathfrak{m}^{p^n}$. Denote the limit $\lim_n r_n \in R$ by $r_\lambda$ since it maps to $\lambda$. It is easy to see that $r_\lambda$ is independent of the choices of $r_n$.

Finally, it suffices to show that $r_\lambda \in R^{p^n}[T]$ for every $n$, so that $r \in K_0$. Since $K = K^{p^n}[\Theta]$, write $\lambda \in K$ as a polynomial

$$\lambda = \sum_{\mu \in \mathcal{F}} c_\mu^{p^n} \mu$$

where $\mathcal{F}$ is some finite set of monomials in $\Theta$. For each $\mu \in \mathcal{F}$ and every $n$, choose $c_{\mu,n} \in R^{p^n}$ that maps to $c_\mu$ modulo $\mathfrak{m}$. The sequence $\{c_{\mu,n}\}_n$ is Cauchy by the same reasoning and converges to $r_{c_\mu}$. Let $\mu'$ be the corresponding monomials in $T$ that lifts $\mu$. Let

$$w_n = \sum_{\mu \in \mathcal{F}} c_{\mu,n}^{p^n} \mu' \in R^{p^n}[T].$$

Since $w_n$ maps to $\lambda$ modulo $\mathfrak{m}$, its limit $r_\lambda = \lim_n w_n \in R^{p^n}[T]$ for every $n$. $\square$

**Remark.** Any coefficient field $K_0$ constructed above contains a lifting $T$ of $\Theta$. Conversely, any coefficient field $L$ containing $T$ is contained in $K_0$. Indeed, the Isomorphism $L \to K$ takes $T$ to $\Theta$, so $T$ is a $p$-basis for $L$. Every element of $L$ is contained in $L^{p^n}[T] \subset R^{p^n}[T]$ for any $n$, so $L \subset K_0$. Therefore, there is a bijection between liftings of the $p$-base $\Theta$ to $R$ and the coefficient fields of $R$. This shows that the choice of coefficient field is never unique in the case when $K$ is imperfect.

# 3 Consequences of CST.

In this section, we will talk about two consequence of the Cohen structure theorem in the equicharacteristic case. Corollary 3.1 gives a clean description of what equicharacteristic complete regular local rings look like up to isomorphism. As a corollary of this, we can prove one direction of a theorem of Kunz that says singularities can be detected using the Frobenius. This theorem has consequence in algebraic geometry.

Let $(R, \mathfrak{m}, K)$ be a complete local ring with coefficient field $K_0 \cong K$. Let elements $f_1, \cdots, f_n \in \mathfrak{m}$ generate an $\mathfrak{m}$-primary ideal. Then by Proposition 1.2, $R$ is finite as a module over $K[[X_1, \cdots, X_n]]$. Now let $d = \dim R$, and $f_1, \cdots, f_d$ be a system of parameters for $R$. Let $\mathcal{K}$ be the kernel of the map $K[[X_1, \cdots, X_d]] \to R$. Then

$$\dim(K[[X_1, \cdots, X_n]]) = d = \dim(R) = \dim(K[[X_1, \cdots, X_d]]/\mathcal{K}).$$

Killing a nonzero prime in a local domain must lower the dimension, so we must have $\mathcal{K} = 0$. This shows that the map $K[[X_1, \cdots, X_d]] \to R$ is injective, so $R$ is a module-finite extension of a formal power series ring. Thus, we may consider a formal power series

$$\sum_{\nu \in \mathbf{Z}_{\geq 0}^d} c_\nu f^\nu$$

where $\nu = (\nu_1, \cdots, \nu_d)$ is a multi-index, $c_\nu \in K$, and $f^\nu = f_1^{\mu_1} \cdots f_d^{\nu_d}$. Since $R$ is complete, this represents an element in $R$. This element is nonzero unless all of the coefficients $c_\nu$ vanish. This fact is sometimes referred to as the *analytic independence of a system of parameters*. Combine this result with the Cohen structure theorem, we have the following super nice description of complete regular local rings.

**Corollary 3.1.** *Let $(R, \mathfrak{m}, K)$ be an equicharacteristic complete regular local ring of dimension $d$. Then $R$ is isomorphic to $K[[X_1, \cdots, X_d]]$.*

More generally, if $R$ is not assumed to be regular, than it is a module-finite extension of a complete regular local ring $S \subset R$.

Now let's talk about Kunz's theorem. Let $R$ be a Noetherian ring of prime characteristic $p > 0$. One can define the Frobenius map $F : r \mapsto r^p$. As an $R$-module, $F_*R$ is the same as $R$ as abelian groups, but with the $R$-module structure defined by $r \cdot x = r^p x$. The following theorem, due to Ernst Kunz, says that singularities can be detected using Frobenius.

**Theorem 3.2** (Kunz)**.** *That $R$ is regular if and only if $F_*R$ is a flat $R$-module.*

Let $X$ be a scheme of prime characteristic $p > 0$ (this means that $\mathcal{O}_X(X)$ has characteristic $p$). One can define a Frobenius map $F$ on such schemes, and say that $X$ is $F$-finite if the Frobenius is a finite morphism of schemes. Kunz's theorem implies that a Noetherian $F$-finite scheme is regular if and only if the coherent sheaf $F_*\mathcal{O}_X$ is locally free. This is because finitely generated flat module over a Noetherian ring is locally free.

In fact, with a bit more work, Kunz's theorem can be used to show that the regular locus of a Noetherian $F$-finite scheme is open. This is a particularly nice geometric result: since Zariski-open sets are heuristically considered to be "very large", so "most" points of such a scheme are regular, which is what one would expect.

The Cohen structure gives a quick proof of one direction in Kunz's theorem. In fact, since both flatness and regularity can be checked locally, we are reduced to the local case. Moreover, since a local Noetherian ring is regular if and only if its completion is regular, and that the Frobenius for a Noetherian local ring is flat if and only if the Frobenius for its completion is flat, we are reduced to the complete local case. Suppose that $R$ is a complete regular local ring. Then Corollary 3.1 allows us to characterize $R = K[[X_1, \cdots, X_d]]$. To see that the Frobenius is flat on $R$, we factor the Frobenius as a composition of three flat extensions:

$$k[[X_1, \cdots, X_d]] \subset k[[X_1^{1/p}, \cdots, X_d^{1/p}]] \subset k^{1/p} \otimes_k k[[X_1^{1/p}, \cdots, X_d^{1/p}]] \subset k^{1/p}[[X_1^{1/p}, \cdots, X_d^{1/p}]].$$

The first extension is flat because the set

$$\{(x_1^{a_1} \cdots x_2^{a_n})^{1/p} \mid 0 \le a_i \le p-1\}$$

forms a free basis for $k[[X_1^{1/p}, \cdots, X_d^{1/p}]]$ over $k[X_1, \cdots, X_d]]$. The second extension is flat because $k^{1/p}$ is a flat $k$-module and flatness is preserved by base change. The third extension is flat because it is a completion. A Composition of flat maps is flat.

The other direction of Kunz's theorem is more difficult, and we refer the readers to [SS] for details.

# 4 Overview of the mixed characteristic case.

We now want a version of the structure theory when the complete local Noetherian ring $R$ need not contain a field. Let $K$ be the residue class field of $R$. If $K$ has characteristic 0, then $\mathbf{Q} \subset R$, so $R$ is equicharacteristic. Thus, it suffices to consider here the case where $K$ has prime characteristic $p > 0$. An example is the ring of $p$-adic integers, or a formal power series ring over such.

Instead of coefficient fields, we now consider so-called coefficient rings.

**Definition.** Let $(R, \mathfrak{m}, K)$ be a complete local ring. A subring $V \subset R$ is called a *coefficient ring* if the natural map it is a complete local ring with maximal ideal $V \cap \mathfrak{m} = pV$, and the natural map $V/pV \to K$ on residue fields is an isomorphism.

If $R$ contains a field, which happens if either $\mathbf{Q} \subset R$ or $pR = 0$, then the coefficient ring degenerates to the coefficient field that we treated before. There are two remaining possibilities:

(i) $p$ is nonzero in $R$ and $p$ is not nilpotent in $R$;

(ii) $p$ is nonzero in $R$ but is nilpotent in $R$.

In the first case, $V$ is a complete DVR with maximal ideal $pV$. In the second case, $V$ is an Artinian local ring with maximal ideal generated by $p$. In fact, in the second case, $V$ has the form $W/p^nW$, where $n \geq 1$ and $W$ is a complete DVR with maximal ideal $pW$.

Like in the equicharacteristic case, the difficult part is to demonstrate the existence of such a coefficient ring $V$. Once that has been done, let $p, u_1, \cdots, u_s$ be generators for $\mathfrak{m}$. Then by the same argument as before (see Section 1), we have that the natural map $V[[X_1, \cdots, X_s]] \to R$ sending $X_i$ to $u_i$ is surjective.

**Theorem 4.1** (Cohen, mixed characteristic)**.** *Every complete local Noetherian ring $R$ that does not contain a field is a homomorphic image of a formal power series ring over a Noetherian discrete valuation ring that maps onto a coefficient ring for $R$.*

I will say a few words on how the existence of a coefficient ring is established, but we will not go into the full detail of the proof. Since we are dealing with the case where $\mathrm{char}(K) = p > 0$, it is natural to consider a $p$-basis for $K/K^p$. The main difficulty here, compared to the equicharacteristic case, is the absence of a naive lift of generators. We are forced to work around that.

The following proposition, which constructs coefficient rings when the maximal ideal of the ring is nilpotent, is the heart of the proof. The general existence of a coefficient ring follows from it with careful analytic manipulations.

**Proposition 4.2.** *Let $(R, \mathfrak{m}, K)$ be a complete local ring with $\mathrm{char}(K) = p > 0$ and $\mathfrak{m}^n = 0$. Let $\Lambda$ be a $p$-basis for $K/K^p$ and choose a lift $T = \{t_\lambda : \lambda \in \Lambda\}$ of $\Lambda$ to $R$. Then $R$ has a* *unique coefficient ring $V$ that contains $T$, constructed as follows. Let $q = p^N$ for $N \geq n-1$, and let $S_N$ be the set of all expressions of the form*

$$\sum_{\mu \in \mathcal{M}} r_\mu^q \mu,$$

*where $\mathcal{M}$ is a finite set of all monomials in $T$ such that the exponent on each element of $T$ is at most $q-1$, and $r_\mu^q \in R^q$. Then we may take*

$$V = S_N + pS_N + p^2 S_N + \cdots + p^{n-1} S_N,$$

*which will be the same as the smallest subring of $R$ containing $R^q$ and $T$.*

It may be helpful to review the proof of Proposition 2.4, as there are similar ideas here.

With some technical work, one will be able to show that for a field $K$ of prime characteristic $p > 0$, there exists a complete Noetherian valuation domain $(V, pV, K)$ with residue class field $K$. (In some old literature, this is called a $p$-ring.) Then, there is, up to isomorphism, a unique coefficient ring of the form $W/p^t W$, where $(W, pW, K)$ is a Noetherian valuation domain. In particular, there is a unique coefficient ring containing a given lifting $T$ to $R$ of a $p$-basis $\Lambda$ of $K$.

**Remark.** The proof strategy outlined here, in both the equicharacteristic and the mixed characteristic case, is due to a paper by Cohen in 1946. For a brief summary of the ideas we recommend [Katz] (beware of different terminology used there).

There is another more modern and arguably cleaner treatment using a notion Grothendieck created called $I$-smoothness, which is obtained by reformulating the theory of nonsingular points in algebraic geometry in terms of an algebraic "infinitesimal analysis" that makes effective use of nilpotent elements. For a proof of the Cohen structure theorem using this machinery, we refer the readers to [Matsumura] and the Stacks Project tag 0323.

# References


[Hoch] Melvin Hochster. *The Structure Theory of Complete Local Rings*. `https://dept.math.lsa.umich.edu/~hochster/615W14/Struct.Compl.pdf`.

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