COFREENESS OF LUBIN-TATE THEORY

YUNHAN (ALEX) SHENG

Abstract. This article is a detailed survey of Section 3 of the paper *The chromatic Nullstellensatz* by Burklund, Schlank, and Yuan [BSY22]. It aims to prove that at the level of algebras, Lubin-Tate theory E is *cofree*, i.e., right adjoint to some forgetful functor forgetting the structure of power operations. This is arguably one of the most technical part of the paper, since it builds upon previous works on power operations in Morava E-theory, most notably by Strickland [Str97], [Str98], Ando-Hopkins-Strickland [AHS04], and Rezk [Rez09]. We shall introduce these relevant results along the way.

Contents

1. Introduction: power operations

Let E be a structured ring spectrum (for instance, a commutative S-algebra). It carries the structure of *power operations* that refines power maps on E^0. To define power operations, let $f : \Sigma^\infty_+ X \to E$ be an element of $E^0(X)$. The symmetric group on m letters Σ_m acts diagonally on the m-fold smash product of f. Taking the homotopy coinvariants of the action gives a map

$$\begin{array}{ccccccc} \Sigma^\infty_+(X^{\times m}_{h\Sigma_m}) & \longrightarrow & (\Sigma^\infty_+ X)^{\wedge m}_{h\Sigma_m} & \xrightarrow{f^{\wedge m}} & E^{\wedge m}_{h\Sigma_m} & \longrightarrow & E \\ \| & & \| & & & & \\ \Sigma^\infty_+(E\Sigma_m \times_{\Sigma_m} X^{\times m}) & & (E\Sigma_m)_+ \wedge_{\Sigma_m} (\Sigma^\infty_+ X)^{\wedge m} & & & & \end{array},$$

where the last map is given by the multiplicative structure on E. The composite map is an element in $E^0(E\Sigma_m \times_{\Sigma_m} X^{\times m})$, so we have produced a map

$$P_m : E^0(X) \to E^0(E\Sigma_m \times_{\Sigma_m} X^{\times m}).$$

Definition. The map P_m constructed above is called the *m-th total power operation* on E. Restricting along the inclusion gives the *m-th power operation*

$$E^0(X) \xrightarrow{P_m} E^0(E\Sigma_m \times_{\Sigma_m} X^{\times m}) \to E^0(X \times B\Sigma_m),$$

which is also denoted by P_m unambiguously. Here, m is referred to as the weight.

The m-th power operation provides a refinement for the map $x \mapsto x^{\times m}$ taking the m-fold power in the following sense:

$$\begin{array}{ccccc} & & E^0(X^m \times_{\Sigma_m} E\Sigma_m) & \xrightarrow{\text{diag}} & E^0(X \times B\Sigma_m) \\ & \overset{P_m}{\nearrow} & \Big\downarrow{\scriptstyle \text{fib}} & & \Big\downarrow{\scriptstyle * \to B\Sigma_m} \\ E^0(X) & \xrightarrow[x \mapsto x^{\times m}]{} & E^0(X^m) & \xrightarrow[\text{diag}]{} & E^0(X) \end{array}.$$

Example (Steenrod operations)**.** Let $E = H\mathbf{Z}/2$ be the mod-2 Eilenberg-Mac Lane spectrum. The associated power operation of weight 2 is

$$P_2 : H^*(X; \mathbf{Z}/2) \to H^*(X \times B\Sigma_2) \cong H^*(X; \mathbf{Z}/2) \otimes (\mathbf{Z}/2)[t].$$

For $x \in H^n(X; \mathbf{Z}/2)$, it is given by the Steenrod squares:

$$P_2(x) = \mathrm{Sq}^n(x) + \mathrm{Sq}^{n-1}(x)t + \cdots + \mathrm{Sq}^0(x)t^n \in H^{2n}(X \times B\Sigma_2).$$

Note that when $t = 0$, we recover the square $P_2(x) = \mathrm{Sq}^n(x) = x^2$. Rezk wrote an illuminating explanation[1] motivating the construction above: one might expect, since cup product is commutative, that taking the square of a cohomology class factors through the quotient X^2/Σ_2. But this isn't quite true, since cup product is only commutative up to the E_∞-structure. So instead, the squaring map should factor through $X \times B\Sigma_2$, after restricting to the diagonal copy.

Example (Adams operations)**.** Let $E = KU$ be the complex K-theory spectrum. According to the splitting principle, a class $[x] \in K^0(X)$ splits as a direct sum of line bundles $x = L_1 \oplus \cdots \oplus L_n$. Taking the m-th power gives a new class $L_1^m \oplus \cdots \oplus L_n^m$. This defines Adams operations $\psi^m : K^0(X) \to K^0(X)$. Since $\psi^p(x) \equiv x^p \bmod p$ for any prime p, there exists an operation $\theta_p : K^0(X) \to K^0(X)$ that *witness* ψ^p as a lift of the Frobenius in the sense that

$$\psi^p(x) = x^p + p\theta_p(x).$$

Note that θ_p satisfy several identities and is called a *p-derivation*. It turns out that one has analogous operations ψ^p and θ_p for any $K(1)$-local E_∞-ring spectrum (fixed at a prime p), such as p-adic K-theory. A generalization of θ_p will be introduced later in Section 5, which turns out to be one of the key ingredients in the proof of our main theorem.

The structure of power operations is captured by the notion of monads. As a motivating example, the homotopy group $\pi_* A$ of a commutative $H\mathbf{F}_p$-algebra A is an algebra over a monad on the category of graded $\mathbf{F}_p$-vector spaces. To be precise, let C be the free monad on $H\mathbf{F}_p$-modules, defined by

$$C(M) = \bigvee_{i \geq 0} M^{\wedge i}/\Sigma_i,$$

so that the algebras over C are precisely commutative $H\mathbf{F}_p$-algebras. The monad C is homotopically well-behaved and descends to a monad (which we still denote by C) on the homotopy category of $H\mathbf{F}_p$-modules, which is equivalent to the category of graded $\mathbf{F}_p$-vector spaces via π_*. On this category, C sends V to the free object on V in the category of graded-commutative $\mathbf{F}_p$-algebras equipped with May-Dyer-Lashof

[1] https://mathoverflow.net/questions/6377/why-does-one-think-to-steenrod-squares-and-powers#:~:text=Steenrod%20operations%20are%20an%20example,%2C%20but%20not%20too%20commutative%22.

operations. Moreover, the natural map $C(\pi_* M) \to \pi_*(CM)$ is an isomorphism. This is proved in the Chapter 9 of [BMMS86].

It is fruitful to study power operations coming from the multiplicative structure of the underlying spectrum in a more general context, such as Morava E-theory, which admits a unique E_∞-structure due to the Goerss-Hopkins-Miller theorem.

Notation. For the remainder of the paper, fix the following data:

- a perfect field k of prime characteristic $p > 0$,
- a formal group $\mathbb{G}_0$ of height N over k,
- the associated Lubin-Tate theory $E(k, \mathbb{G}_0)$ will be denoted by $E(k)$.

Let A be a perfect k-algebra. Then we write $E(A)$ for the $E(k)$-algebra $E(A, (\mathbb{G}_0)_A)$. Then we have[2]

$$\pi_0 E(A) \cong W(A)[[u_1, \cdots, u_{N-1}]].$$

In [Rez09], Rezk constructs a monad $\mathbb{T}$ on E_*-modules that models the algebraic theory of power operations on $K(N)$-localized Morava E-theory E. We shall go over his construction and summarize the properties of $\mathbb{T}$ in Section 2. The reader will notice many similarities between Rezk's results and classical results in [BMMS86]. Consequently, there is a forgetful functor $U_{\mathbb{T}}$ from $\mathbb{T}$-algebras $\mathrm{Alg}_{\mathbb{T}}$ to E_0-algebras forgetting the structure of power operations. Let $\mathfrak{m}$ be the Landweber ideal $(p, u_1, \cdots, u_{N-1})$ and Perf_k be the category of perfect k-algebras. Let $(-)^{\#}$ be the colimit perfection which is left adjoint of the inclusion $\mathrm{Perf}_k \hookrightarrow \mathrm{Alg}_k$. Explicitly, it is given by taking the colimit of consecutive Frobenius maps. The main goal of this paper is to prove the following adjunction.

Theorem 1.1 ([BSY22], 3.4)**.** *There is an adjunction*

$$(U_{\mathbb{T}}(-)/\mathfrak{m})^{\sharp} : \mathrm{Alg}_{\mathbb{T}} \leftrightarrows \mathrm{Perf}_k : \pi_0 E(-),$$

where the right adjoint $\pi_0 E$ is fully faithful.

In other words, at the level of algebras, roughly speaking, Lubin-Tate theory E is *cofree*, being right adjoint to some variant of the forgetful functor. By realizing it as right adjoint, it is easier to construct maps *into* Lubin-Tate theories, to which a large portion of [BSY22] is devoted.

We give an outline of the proof as well as the organization of this survey. Firstly, we will show that it suffices to prove that a certain reduced unit map

$$\overline{\mathrm{ev}}_A : \pi_0 E(A) \to W_{\mathbb{T}}(A)$$

is an isomorphism for any perfect k-algebra A. This is achieved by investigating Rezk's algebraic approximation functor $\mathbb{T}$, which is explained in Section 2. Then, using the Witt filtration introduced in Section 3, we further reduce it to showing

$$\overline{\mathrm{ev}}_k : \pi_0 E(k) \to W_{\mathbb{T}}(k)$$

is an isomorphism. In Section 4 we interpret power operations as moduli problem using deformation theory. This is crucial in proving the surjectivity of $\overline{\mathrm{ev}}_k$. Finally, in Section 5 we introduce the notion of p-derivation and conclude the proof.

[2] This is a well-known folklore, and a proof is given in [Lur18].

Remark 1.2. We will not distinguish between $T(N)$-local spectra and $K(N)$-local ones since we are working with ring spectra. It is a consequence of the nilpotence theorem that for a ring spectrum R, $R \wedge T(N) \simeq 0$ if and only if $R \wedge K(N) \simeq 0$. The reader will notice that while Rezk works in the $K(N)$-local category in [Rez09], many results cited in [BSY22] are stated $T(N)$-locally.

2. Rezk's algebraic approximation functor

In this section, we review the construction by [Rez09] of a monad $\mathbb{T}$ on E_*-modules that mimics the aforementioned monad C on graded $\mathbf{F}_p$-vector spaces. Roughly speaking, this monad captures information of power operations in E.

We say that a E-module M is finitely generated (resp. finite free) if $\pi_* M$ is finitely generated (resp. finitely generated and free) over E_*. A E-module M is finite free if and only if M is equivalent to a finite wedge $\vee_{i=1}^k \Sigma^{d_i} E$. Let $\mathrm{Mod}_E^{\mathrm{ff}}$ denote the full subcategory of Mod_E consisting of finite free E-modules.

Proposition 2.1. *The functor $\pi_* : h\mathrm{Mod}_E^{\mathrm{ff}} \to \mathrm{Mod}_{E_*}^{\mathrm{ff}}$ is an equivalence.*

Definition. The *n-th extended power* $\mathbb{P}_n : \mathrm{Mod}_E \to \mathrm{Mod}_E$ is a functor given by

$$\mathbb{P}_n(M) := (M^{\wedge_E n})_{h\Sigma_n} = (E\Sigma_n) \wedge_{\Sigma_n} M^{\wedge_E n}$$

The *free E-algebra functor* $\mathbb{P} : \mathrm{Mod}_E \to \mathrm{Mod}_E$ is the functor $\mathbb{P} := \vee_{n \geq 0} \mathbb{P}_n$.

Note that $\mathbb{P}$ defines a monad on the homotopy category $h\mathrm{Mod}_E$, and commutative E-algebras are exactly algebras over this monad. Write L_K for the Bousfield localization functor with respect to the Morava K-theory $K = K(N)$.

Proposition 2.2. *The functor $\mathbb{P}_n$ preserves K-homology isomorphisms. In particular, the natural transformation*

$$L_K \mathbb{P}_n(j) : L_K \mathbb{P}_n \to L_K \mathbb{P}_n L_K$$

is an isomorphism for $j : M \to L_K M$. The functor $L_K \mathbb{P} : h\mathrm{Mod}_E \to h\mathrm{Mod}_E$ admits a unique monad structure, and j induces a map $\mathbb{P} \to L_K \mathbb{P}$ of monads.

Definition. The functor $\hat{\mathbb{P}} := L_K \mathbb{P}$ is called the *completed* free algebra functor.

In general, $\mathbb{P}_n$ does not preserve finite free E-modules, but $L_K \mathbb{P}_n$ does.

Proposition 2.3 ([Rez09], 3.17)**.** *If M is finite free, then so is $L_K \mathbb{P}_n(M)$.*

Consider the following diagram of categories

$$\begin{array}{ccccc} h\mathrm{Mod}_E^{\mathrm{ff}} & \xhookrightarrow{i} & h\mathrm{Mod}_E & \xrightarrow{L_K \mathbb{P}_n} & h\mathrm{Mod}_E \\ {\scriptstyle \pi_*}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ \mathrm{Mod}_{E_*}^{\mathrm{ff}} & \xhookrightarrow{j} & \mathrm{Mod}_{E_*} & \overset{}{\underset{\mathbb{T}_n}{\dashrightarrow}} & \mathrm{Mod}_{E_*} \end{array},$$

where the square on the left commutes strictly.

Definition. The *algebraic approximation functor* $\mathbb{T}_n : \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$ is defined to be the left Kan extension of the functor $\pi_* L_K \mathbb{P}_n i$ along $\pi_* i = j \pi_*$. Define $\mathbb{T} : \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$ to be the sum $\oplus_{n \geq 0} \mathbb{T}_n$.

Note that the left Kan extension exists because $h\mathrm{Mod}_E^{\mathrm{ff}}$ is essentially small and Mod_{E_*} is cocomplete. By the universal property of left Kan extensions, there is a natural transformation

$$\alpha_n : \mathbb{T}_n(\pi_* M) \to \pi_*(L_K \mathbb{P}_n M).$$

Again, let $\alpha : \mathbb{T}(\pi_* M) \to \pi_*(\hat{\mathbb{P}} M)$ be the sum $\oplus_{n\geq 0}\alpha_n$.

Proposition 2.4 ([Rez09], 4.8)**.** *If M is finite free, then α_n is an isomorphism.*

This explains the nomenclature of $\mathbb{T}_n$: the monad *approximates* the algebraic structure (i.e., the structure of power operations) of $L_K\mathbb{P}_n$.

Proposition 2.5 ([Rez09], 4.10)**.** *The functor $\mathbb{T}$ admits the structure of a monad compatible with the monad structure on $\hat{\mathbb{P}}$ in the sense that the diagrams*

$$\begin{array}{ccc} \pi_* M & \longrightarrow & \mathbb{T}(\pi_* M) \\ & \searrow & \downarrow{\scriptstyle \alpha} \\ & & \pi_* L_K \mathbb{P}(M) \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathbb{T}\mathbb{T}(\pi_* M) & \longrightarrow & \mathbb{T}(\pi_* M) \\ {\scriptstyle \alpha\mathbb{T}\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \pi_*\hat{\mathbb{P}}\hat{\mathbb{P}}(M) & \longrightarrow & \pi_*\hat{\mathbb{P}}(M) \end{array}$$

commute, where the unlabeled maps are given by the structure maps of the monads.

Moreover, Rezk proved that the $\mathbb{T}_n$'s are symmetric monoidal and preserve finite free E_*-modules.

Now we wish to understand algebras $\mathrm{Alg}^*_{\mathbb{T}}$ over the monad $\mathbb{T}$. Every $\mathbb{T}$-algebra is a graded-commutative E_*-algebra, so there is a forgetful functor $U_{\mathbb{T}} : \mathrm{Alg}^*_{\mathbb{T}} \to \mathrm{Alg}_{E_*}$.

Definition. Let $(\mathcal{C}, \otimes)$ be an abelian tensor category and $\mathcal{A}$ be the category of commutative monoids in $\mathcal{C}$ with respect to $\otimes$. A functor $U : \mathcal{D} \to \mathcal{A}$ is *plethyistic* if it reflects isomorphisms and admits both a left adjoint F and a right adjoint G.

By Beck's monadicity theorem, $\mathcal{D}$ is equivalent to the category of algebras (resp. coalgebras) over the monad UF (resp. comonad UG).

Proposition 2.6 ([Rez09], 4.19, 4.23)**.** *The forgetful functor $U_{\mathbb{T}} : \mathrm{Alg}^*_{\mathbb{T}} \to \mathrm{Alg}_{E_*}$ is plethyistic and commutes with colimits.*

Remark 2.7. There is an L-complete version of the above construction. Recall that E_0 is a complete local ring with maximal ideal $\mathfrak{m} = (p, u_1, \cdots, u_{N-1})$, and a E_*-module M is *L-complete* in the sense of Greenless and May [GM92] if the map $L_0 M \to M^\wedge_{\mathfrak{m}}$ is an isomorphism. Here L_0 is the 0-th left derived functor of the $\mathfrak{m}$-adic completion $(-)^\wedge_{\mathfrak{m}}$, which is neither left nor right exact viewed as a functor on Mod_{E_*}. Rezk defined the *completed* approximation functors $\mathbb{T}^\wedge_n$ on the full subcategory of L-complete E_*-modules and proved that the natural comparison map $\mathbb{T}^\wedge_n(\pi_* M) \to \pi_*(L_K\mathbb{P}_n M)$ is an isomorphism if M is flat. Barthel and Frankland [BF15] further showed that $\mathbb{T}^\wedge_n$ is a monad and the forgetful functor $U^\wedge_{\mathbb{T}} : \mathrm{Alg}^*_{\mathbb{T}^\wedge} \to \mathrm{Alg}_{E_*}$ is plethyistic. The completed version approximates better since it takes into account the fact that $\pi_* L_K \mathbb{P} M$ is L-complete.

Therefore, $U_{\mathbb{T}}$ admits both a left adjoint $F_{\mathbb{T}}$ and a right adjoint $W_{\mathbb{T}}$:

$$\mathrm{Alg}^*_{\mathbb{T}} \underset{W_{\mathbb{T}}}{\overset{F_{\mathbb{T}}}{\leftleftarrows}} \!\!\xrightarrow{\;U_{\mathbb{T}}\;} \mathrm{Alg}_{E_*}\ ,$$

where $W_{\mathbb{T}}$ is explicitly given by

$$W_{\mathbb{T}}(-) = \mathrm{Hom}_{\mathrm{Alg}_{E_*}}(\mathbb{T}(E_0), -). \tag{2.8}$$

In other words, an element of $W_{\mathbb{T}}(B)$ can be thought of as a B-valued functional on the module of power operations. The unit $\mathrm{ev}_B : B \to W_{\mathbb{T}} U_{\mathbb{T}}(B)$ of the adjunction sends $b \in B$ to the functional which evaluates power operations on b. It can alternatively be thought of as a universal lift of the total power operation on B to a $\mathbb{T}$-*algebra* map, in the following sense: for a $K(N)$-local commutative E-algebra R, composing $\mathrm{ev}_{\pi_0 R} : \pi_0 R \to W_{\mathbb{T}}(\pi_0(R))$ with the embedding

$$W_{\mathbb{T}}(\pi_0(R)) \cong \mathrm{Hom}_{\mathrm{Alg}_{E_*}}(\mathbb{T}(E_0), \pi_0 R) \hookrightarrow \mathrm{Hom}_{\mathrm{Mod}_{E_*}}(\mathbb{T}(E_0), \pi_0 R) \cong \prod_i R^0(B\Sigma_i)$$

gives a map $\pi_0 R \to \prod_i R^0(B\Sigma_i)$ that can be identified with the product of power operations. Furthermore, define the *reduced* evaluation $\overline{\mathrm{ev}}_A$ to be as follows:

$$\begin{array}{ccc} & W_{\mathbb{T}} U_{\mathbb{T}} \pi_0 E(A) & \\ {}^{\mathrm{ev}_A}\nearrow & & \searrow^{W_{\mathbb{T}}(-/\mathfrak{m})} \\ \pi_0 E(A) & \xrightarrow[\overline{\mathrm{ev}}_A]{} & W_{\mathbb{T}}(A) \end{array} .$$

As mentioned in the introduction, to prove Theorem 1.1 it amounts to showing that $\overline{\mathrm{ev}}_A$ is an isomorphism. Indeed, consider the following series of adjunctions:

$$\mathrm{Alg}_{\mathbb{T}} \underset{W_{\mathbb{T}}}{\overset{U_{\mathbb{T}}}{\rightleftarrows}} \mathrm{CRing}_{E_0} \rightleftarrows \mathrm{CRing}_k \underset{}{\overset{(-)^{\#}}{\rightleftarrows}} \mathrm{Perf}_k \ .$$

If $\overline{\mathrm{ev}}_A$ is an isomorphism, then the right adjoint above is naturally equivalent to the functor $\pi_0 E(-)$, which is precisely the desired adjunction of Theorem 1.1. The fully faithfulness follows from the fact that the counit map $(\pi_0 E(A)/\mathfrak{m})^{\#} \to A$ is an isomorphism. From now on we will focus on proving that $\overline{\mathrm{ev}}_A$ is an isomorphism.

Example. If height $N = 1$, then $E = E(\mathbf{F}_p, \mathbb{G}_m) = KU_p$ is the p-adic K-theory, and $\mathrm{Alg}_{\mathbb{T}}^*$ can be identified with the category CRing_δ of δ-rings. The p-typical Witt vectors $W = W_{\mathbb{T}} : \mathrm{Cring} \to \mathrm{CRing}_\delta$ is right adjoint to the forgetful functor $U_{\mathbb{T}} = U : \mathrm{CRing}_\delta \to \mathrm{CRing}$. This explains why we use the letter W for the right adjoint. Note that $W(A) = \pi_0 E(A)$, so that composing with the adjunction

$$(-/p)^\sharp : \mathrm{CRing} \leftrightarrows \mathrm{Perf}_{\mathbf{F}_p} : \mathrm{incl}$$

gives an adjunction

$$(U(-)/p)^\sharp : \mathrm{CRing}_\delta \leftrightarrows \mathrm{Perf}_{\mathbf{F}_p} : \pi_0 E(-).$$

This provides evidence for Theorem 1.1 to be true.

3. The Witt filtration

In this section we shall further reduce Theorem 1.1 to proving that

$$\overline{\mathrm{ev}}_k : \pi_0 E(k) \to W_{\mathbb{T}}(A)$$

is an isomorphism. The tool we use for this reductive step is the Witt filtration, which is a filtration on $W_{\mathbb{T}}$ induced, via the identification (2.8), by the weight grading on the monad $\mathbb{T} = \oplus_j \mathbb{T}_j$. To be precise, note that any power operation $\lambda \in \mathbb{T}(E_0)$ determines, via evaluation, a functional

$$\lambda^* : W_{\mathbb{T}}(A) \cong \mathrm{Hom}_{\mathrm{Alg}_{E_*}}(\mathbb{T}(E_0), A) \to A$$

It is easy to verify that the subset

$$W_{\mathbb{T}}^{\geq r}(A) := \{v \in W_{\mathbb{T}}(A) \mid \lambda^* v = 0 \text{ for any } \lambda \in \mathbb{T}(E_0) \text{ of weight less than } p^r\}$$

of $W_{\mathbb{T}}(A)$ is an ideal.

Definition. Define $W_{\mathbb{T}}^{\leq r}(A) := W_{\mathbb{T}}(A)/W_{\mathbb{T}}^{\geq r+1}(A)$. The *Witt filtration* on $W_{\mathbb{T}}(A)$ is the filtration

$$W_{\mathbb{T}}(A) \to \cdots \to W_{\mathbb{T}}^{\leq 2}(A) \to W_{\mathbb{T}}^{\leq 1}(A) \to W_{\mathbb{T}}^{\leq 0}(A) \cong A.$$

The Witt filtration behaves well with respect to the additive power operations:

Proposition 3.1 ([BSY22], 3.24)**.** *The r-th total additive power operation*

$$P_r/I_{\mathrm{tr}} : E^0 \to E^0(B\Sigma_{p^r})/I_{\mathrm{tr}}$$

factors, as a ring map, through the composite $E^0 \xrightarrow{\mathrm{ev}_{E^0}} W_{\mathbb{T}}(E_0) \to W_{\mathbb{T}}^{\leq r}(E_0)$.

Viewing $\mathbb{T}(E_0)$ as a polynomial E_0-algebra, Strickland computed ([Str98], 3.6) that the E_0-module of indecomposables of weight p^r is free of rank

$$\bar{d}(r) = \prod_{j=1}^{N-1} \frac{p^{r+1}-1}{p^j - 1},$$

where $\bar{d}(r)$ counts the number of subgroups of $(\mathbf{Q}_p/\mathbf{Z}_p)^N$ of cardinality p^r. So we obtain a non-canonical identification

$$W_{\mathbb{T}}(A) \cong \mathrm{Hom}_{\mathrm{Alg}_{E_*}}(\mathbb{T}(E_0), A) \cong \prod_{r \geq 0} A^{\bar{d}(r)}.$$

It turns out that this identification also behaves well with respect to the Witt filtration, in the following sense.

Proposition 3.2 ([BSY22], 3.27)**.** *For $A \in \mathrm{Alg}_{E_0}$, there is a natural isomorphism of abelian groups*

$$W_{\mathbb{T}}^{=r} := W_{\mathbb{T}}^{\geq r}(A)/W_{\mathbb{T}}^{\geq r+1}(A) \cong A^{\bar{d}(r)}$$

that fits into the following commutative diagram of sets:

$$\begin{array}{ccc} W_{\mathbb{T}}^{\geq r}(A) & \longrightarrow & W_{\mathbb{T}}^{=r}(A) \\ \cong\downarrow & & \downarrow\cong \\ \prod_{i \geq r} A^{\bar{d}(i)} & \longrightarrow & A^{\bar{d}(r)} \end{array}$$

Since $\pi_0 E(A) \cong W(A)[[u_1, \cdots, u_{N-1}]]$, we can view $W_{\mathbb{T}}(A)$ as a $W(A)$-algebra. For $a \in A$, let $[a]$ denote its lift in $W(A)$. Then for $v \in W_{\mathbb{T}}(A)$ and $\lambda \in \mathbb{T}(E_0)_{p^r}$, the A-module structure is given by $\lambda^*([a] \cdot v) = a^{p^r} \lambda^* v$ ([BSY22], 3.28). This A-module structure descends to the associated graded of the filtration, so we have:

Proposition 3.3 ([BSY22], 3.29)**.** *As an A-module, $W_{\mathbb{T}}^{=r}(A)$ is free of rank $\bar{d}(r)$.*

Now we are ready to prove the reduction step.

Theorem 3.4 ([BSY22], 3.31). *If $\overline{\mathrm{ev}}_k : \pi_0 E(k) \to W_{\mathbb{T}}(k)$ is an isomorphism, then $\overline{\mathrm{ev}}_A : \pi_0 E(A) \to W_{\mathbb{T}}(A)$ is an isomorphism for any perfect k-algebra A.*

The proof is explained in detail in the original paper and we refrain from copying it verbatim here. To summarize it from a different angle, consider the diagram

$$\begin{array}{ccc} \pi_0 E(k) \otimes_{W(k)} W(A) & \xrightarrow{\overline{\mathrm{ev}}_k \otimes 1} & W_{\mathbb{T}}(k) \otimes_{W(k)} W(A) \\ \downarrow & & \downarrow \\ \pi_0 E(A) & \xrightarrow{\overline{\mathrm{ev}}_A} & W_{\mathbb{T}}(A) \end{array} .$$

Our previous analysis shows that the right-hand map is an isomorphism on the associated graded and hence also after the completion with respect to the filtration. The left-hand map is an isomorphism after $\mathfrak{m}$-adic completion. Finally, the theorem follows from comparing these two filtrations, i.e., the $\mathfrak{m}$-adic topology and $W_{\mathbb{T}}$-adic topology conincide on $\pi_0 E(k)$.

4. Moduli interepretation of power operations

One of the key inputs in the proof of our main result is the following.

Theorem 4.1 ([BSY22], 3.13). *The mod-$\mathfrak{m}$ total additive power operation*

$$\overline{P}_r : E^0 \to E^0(B\Sigma_{p^r})/(I_{\mathrm{tr}}, \mathfrak{m})$$

is surjective for any $r \geq 0$.

The entire Section 3.4 *Rigidity of isogenies* of [BSY22] is devoted to proving this result. To understand the whole picture requires background knowledge on the moduli-theoretic interpretation of power operations in Morava E-theory. A survey paper that does a great job explaining is [Stap], and for a more concise summary see the writeup [YS] taken by the author while reading the aforementioned survey.

To briefly summarize the result used here , let A be a finite abelian group and consider the r-th power operation $P_r : E^0(BA) \to E^0(BA \times B\Sigma_r)$. Strickland ([Str97]) proved that $E^0(B\Sigma_r)$ is a free E^0-module, so that by the Künneth formula P_r takes the form

$$P_r : E^0(BA) \to E^0(BA) \otimes_{E^0} E^0(B\Sigma_r).$$

This map is multiplicative but *not* additive. There is a smallest ideal $I_{\mathrm{tr}} \subset E^0(B\Sigma_r)$ called the *transfer ideal* such that the quotient

$$P_r/I_{\mathrm{tr}} : E^0(BA) \to E^0(BA) \otimes_{E^0} E^0(B\Sigma_r)/I_{\mathrm{tr}}$$

is additive. We call the ring homomorphism P_r/I_{tr} the *additive* power operation.

In the adjacent world of algebraic geometry, there are so-called *moduli problems*, which are functors that associate to each object X in the source category a collection of geometric objects parametrized by X. We say that this moduli problem has a *solution* if the functor is (co)representable. For instance, the functor

$$\mathrm{LT} : \text{complete local rings} \to \text{groupoids}$$

that sends a complete local ring $(R, \mathfrak{m})$ to the groupoid of deformations of $\mathbb{G}_0/k$ to $(R, \mathfrak{m})$ is, by the Lubin-Tate theorem, corepresented by a complete local ring $\mathcal{O}_{\mathrm{LT}}$

non-canonically isomorphic to $W(k)[[u_1, \cdots, u_{N-1}]]$. Likewise, one can form other moduli problems

$$\mathrm{Hom}(A, \mathbb{G}_u),\ \mathrm{Sub}_{p^r}(\mathbb{G}_u) : \text{complete local rings} \to \text{groupoids}$$

that encode extra data that builds upon the Lubin-Tate moduli problem. We leave the detailed definition of these moduli problems to the reference given above. We only need to know that their corepresenting rings are finitely generated and free as modules over the Lubin-Tate ring $\mathcal{O}_{\mathrm{LT}}$.

Hopkins, Kuhn, and Ravenel ([HKR00]) proved that there is a canonical isomorphism between $E^0(BA)$ and the corepresenting ring of $\mathrm{Hom}(A, \mathbb{G}_u)$, and Strickland ([Str98]) proved that $E^0(B\Sigma_r)/I_{\mathrm{tr}}$ is canonically isomorphic to the corepresenting ring of $\mathrm{Sub}_{p^r}(\mathbb{G}_u)$. The seminal work of Ando, Hopkins, and Strickland ([AHS04]) showed that the additive r-th power operation corresponds to the ring of functions on the map of moduli problems

$$\mathrm{Sub}_{p^r}(\mathbb{G}_u) \otimes_{\mathrm{LT}} \mathrm{Hom}(A^\vee, \mathbb{G}_u) \to \mathrm{Hom}(A^\vee, \mathbb{G}_u).$$

Let $\phi^r : \mathbb{G}_0 \to \mathbb{G}_0^{(r)}$ be the r-fold relative Frobenius isogeny. Then $\mathrm{Sub}_{p^r}(\mathbb{G}_u)$ can be identified with the deformation $\mathrm{Def}(\phi^r)$ classifying deformations of ϕ^r. In our case A is trivial, so $\mathrm{Hom}(A^\vee, \mathbb{G}_u)$ degenerates to $\mathrm{Def}(\mathbb{G}_0^{(r)})$. Moreover, since k is a perfect $\mathbf{F}_p$-algebra, it can be shown that any complete local ring $k \to B/\mathfrak{m}_B$ lifts uniquely to a map $W(k) \to B$. Combining all these together we obtain a commutative diagram

$$\begin{array}{ccccc} \mathrm{Def}(\phi^r) & \xrightarrow{\sigma} & \mathrm{Def}(\mathbb{G}_0) & \longleftarrow & \mathrm{Spf}(k) \\ {\scriptstyle\tau}\downarrow & & \downarrow & \swarrow & \\ \mathrm{Def}(\mathbb{G}_0^{(r)}) & \longrightarrow & \mathrm{Spf}(W(k)) & & \end{array} \tag{4.2}$$

,

where σ and τ remembers the source and target, respectively. Therefore, the mod-$\mathfrak{m}$ version $\overline{P}_r$ corresponds to the map

$$\begin{aligned} \mathrm{Spf}(k) \times_{\mathrm{Def}(\mathbb{G}_0)}{}^{\sigma} \mathrm{Def}(\phi^r) &\to \mathrm{Spf}(k) \times_{\mathrm{Spf}(W(k))} \mathrm{Def}(\phi^r) \\ &\xrightarrow{\overline{P}_r} \mathrm{Spf}(k) \times_{\mathrm{Spf}(W(k))} \mathrm{Def}(\mathbb{G}_0^{(r)}) \\ &\cong \mathrm{Spf}(k) \times_{\mathrm{Spf}(W(k))} \mathrm{Def}(\mathbb{G}_0) \to \mathrm{Def}(\mathbb{G}_0). \end{aligned}$$

The last map is a closed immersion. The composite of first two maps comes from the left-hand square of (4.2) base changed along the map $\mathrm{Spf}(k) \to \mathrm{Spf}(W(k))$. More precisely, it is obtained from taking the fiber product of the map

$$(\sigma, \tau) : \mathrm{Def}(\phi^r) \to \mathrm{Def}(\mathbb{G}_0^{(r)}) \times_{\mathrm{Spf}(W(k))} \mathrm{Def}(\mathbb{G}_0)$$

with $\mathrm{Spf}(k)$ over $\mathrm{Spf}(W(k))$ and then viewing everything as over $\mathrm{Def}(\mathbb{G}_0)$. Thus we have reduced Theorem 4.1 to showing that the map on the ring of functions of (σ, τ) is surjective. The upshot is that we have turned our original problem in topology into a problem in algebraic geometry, and now we have tools from deformation theory at our disposal.

Lemma 4.3 ([BSY22], 3.40)**.** *Let B be a complete local ring with $p = 0$ in B. Let $\mathbb{G}$ and $\mathbb{G}'$ be formal groups over B of height $n \geq 1$ and let $\psi_0 : \pi^*\mathbb{G} \to \pi^*\mathbb{G}'$ be the map pulled back along the special fibers $\pi : B \to B/\mathfrak{m}_B$. Then there is at most one map $\psi : \mathbb{G} \to \mathbb{G}'$ such that $\pi^*\psi = \psi_0$.*

Proof. Suppose $\psi_0 = 0$. It suffices to show that the map ψ extending ψ_0 is likewise zero. After choosing local coordinates, the map $\psi : \mathbb{G} \to \mathbb{G}'$ corresponds to a map $\mathcal{O}_{\mathbb{G}'} \cong B[[y]] \to B[[x]] \cong \mathcal{O}_{\mathbb{G}}$, which is determined by the image $y \mapsto g(x)$. By assumption $g(x) \equiv 0 \mod \mathfrak{m}_B$. Suppose ψ is nonzero, then here exists a j such that all coefficients of g are in $\mathfrak{m}_B^j$, so that $g([p](x)) = p \cdot g(x) \equiv 0 \mod \mathfrak{m}_B^{j+1}$. On the other hand, there exists an i such that $g(x) \equiv ax^i \mod (\mathfrak{m}_B^{j+1}, x^{i+1})$ for some $a \in \mathfrak{m}_B^j$. Since $\mathbb{G}$ is of height n, we have

$$g([p]x) \cong a(u_n x^{p^n})^i \mod (\mathfrak{m}_B^{j+1}, x^{i+1})$$

for some unit u_n. This contradicts with $g([p]x) \equiv 0 \mod \mathfrak{m}_B^{j+1}$. Hence $\psi = 0$. □

Remark 4.4. This is what the original authors meant by the *rigidity of isogenies*: there is only one way of extending the isogeny over the special fibers.

Proposition 4.5 ([BSY22], 3.39)**.** *Let $q_0 : \mathbb{G}_0 \to \mathbb{G}_0'$ be an isogeny of formal groups over k. Then the map of schemes*

$$(\sigma, \tau) : \mathrm{Def}(q_0) \to \mathrm{Def}(\mathbb{G}_0) \times_{\mathrm{Spf}(W(k))} \mathrm{Def}(\mathbb{G}_0')$$

is a closed immersion.

Proof. Since (σ, τ) corresponds to a map of complete local rings, it suffices to show that the corresponding map is surjective modulo the maximal ideal. Geometrically, this corresponds to showing that the map

$$\Theta : \mathrm{Spf}(k) \times_{\mathrm{Def}(\mathbb{G}_0)} {}^{\sigma}\mathrm{Def}(q_0)^{\tau} \times_{\mathrm{Def}(\mathbb{G}_0')} \mathrm{Spf}(k) \to \mathrm{Spf}(k)$$

of schemes is a closed immersion. Here, the map $\mathrm{Spf}(k) \to \mathrm{Def}(\mathbb{G}_0)$ classifies sending a k-algebra B to the trivial deformations of $\mathbb{G}_0$ to B (and similarly for $\mathbb{G}_0'$). Thus, the left-hand side classifies deformations of q_0 where the target and source are trivial deformations. Therefore, we have found an isogeny $a : f^*\mathbb{G}_0 \to f^*\mathbb{G}_0'$ for every $f : k \to B$ such that $\pi^* q = q_0$ over the special fiber. By Lemma 4.3, there is exactly one of extending q_0 to such a q, which means that Θ is an isomorphism. □

5. Using the p-derivation

In this section, we put in place the last piece of the puzzle, the p-derivation, and use it to complete the proof of our main theorem, which has already been reduced to showing that $\overline{\mathrm{ev}}_k : \pi_0 E(k) \to W_{\mathbb{T}}(k)$ is an isomorphism.

Definition. A *p-derivation* is an operation $\theta \in \mathbb{T}(E_0)_p$ of weight p such that for any $\mathbb{T}$-algebra B and $x, y \in B$, we have

$$\theta(x+y) = \theta(x) + \theta(y) + \frac{1}{p}(x^p + y^p - (x+y)^p).$$

The existence and construction of p-derivations can be found in the works of Rezk or Mathew, Naumann, and Noel [MNN15]. Note that θ is a non-additive power operation, but it *is* additive after reduction:

$$\theta(x+y) \equiv \theta(x) + \theta(y) \mod (xy).$$

The crucial property that we make use of the p-derivation is

$$\theta^j(p^k x + p^{k+1}(\cdots)) \equiv p^{k-j} x^{p^j} \mod p^{k-j+1}. \tag{5.1}$$

Recall that $\overline{\mathrm{ev}}_k$ can be identified with the total mod-$\mathfrak{m}$ additive power operations $\overline{P}_r : E^0 \to E^0(B\Sigma_{p^r})/(I_{\mathrm{tr}}, \mathfrak{m})$. We now prove that it is injective.

Lemma 5.2 ([BSY22], 3.45(1))**.** *Let $x \in E_0$ be an element such that $\overline{P}_j(x) \neq 0$ for $1 \leq j \leq n$. There exists an element $y \in E_0$ such that the image of $p^i x + p^{i+1} y$ for each i under the composite*

$$E^0 \xrightarrow{\overline{\mathrm{ev}}_k} W_{\mathbb{T}}(k) \to W_{\mathbb{T}}^{\leq i+j}(k)$$

is nonzero.

Proof. Since $\overline{P}_j(x) \neq 0$, there exists an additive power operation $Q \in \mathbb{T}(E_0)$ of weight p^j such that $\overline{Qx} \neq 0$. Consider the operation $\theta^i Q$. By (5.1) we have

$$\theta^i Q(p^i x + p^{i+1} y) = \theta^i(p^i Q(x) + p^{i+1} Q(y)) \equiv Q(x)^{p^i} \not\equiv 0 \mod p.$$

Since $\theta^i Q$ has weight p^{i+j}, $p^i x + p^{i+1} y$ is indeed detected in $W_{\mathbb{T}}^{\leq i+j}(k)$. □

Lemma 5.3 ([BSY22], 3.32)**.** *Let $x \in E_0$ be such that $x \not\equiv 0$ mod p. There exists an $r > 0$ such that $\overline{P}_r(x) \not\equiv 0$ mod $\mathfrak{m}$.*

Proof. See Section 3.3 of [BSY22]. This relies on a transchromatic induction that is at the heart of the whole paper. It is explained in detail in Section 5 of [BSY22]. □

Theorem 5.4. *The total mod-$\mathfrak{m}$ additive power operation $\overline{P}_r$ is injective.*

Proof. We prove the theorem by showing that for any $y \in E_0$, there is an additive power operation Q and some i such that the mod-$\mathfrak{m}$ reduction of $\theta^i Q(y)$ is nonzero. Write $y = p^i z$ for some $z \not\equiv 0 \mod p$. By Lemma 5.3 there exists an additive power operation Q of weight p^j such that $Qz \not\equiv 0 \mod \mathfrak{m}$. By (5.1) we have $\theta^i Q(y) \equiv Q(z)^{p^i} \mod p$, which is nonzero mod $\mathfrak{m}$ since $\overline{Qz} \in k$. □

The surjectivity of $\overline{\mathrm{ev}}_k$ is slightly trickier. Since the $\mathfrak{m}$-adic topology on $E(k)$ is the finest topology induced by a collection of maps out to Artinian $W(k)$-algebras, it suffices to show that $\pi_0 E(k) \to W_{\mathbb{T}}^{\leq r}(k)$ is surjective for each r. This will be done by a clever dimension counting argument. We need the following lemma.

Lemma 5.5. *Let $\mathcal{O}$ be a discrete valuation ring with uniformizer π and residue field $\kappa = \mathcal{O}/\pi$. Let $V_0, V_1, \cdots, V_r$ be a collection of finite dimensional κ-vector spaces and $\mathcal{E}$ a finitely generated $\mathcal{O}$-module equipped with surjections $f_i : \mathcal{E}/\pi\mathcal{E} \to V_i$ of κ-vector spaces for $0 \leq i \leq r$ satisfying the following condition: if $x \in \mathcal{E}$ is such that $f_i(x) \neq 0$ for all i, then for any $y \in \mathcal{E}$ we have $\pi^i x + \pi^{i+1} y \neq 0$. Then we have the following bound on the $\mathcal{O}$-module length of $\mathcal{E}$:*

$$\mathrm{len}_{\mathcal{O}}(\mathcal{E}) \geq \sum_{i=0}^{r} \dim_\kappa(V_i).$$

Proof. Let $\mathrm{gr}_i = \pi^i \mathcal{E}/\pi^{i+1}\mathcal{E}$ denote the i-th piece of the associated graded of the π-adic filtration on $\mathcal{E}$. Now choose sections $g_i : V_i \to \mathcal{E}/\pi\mathcal{E}$ of the surjections f_i. The condition on f_i implies that $\pi^i g_i : V_i \to \mathrm{gr}_i$ is injective, so that we have

$$\mathrm{len}_{\mathcal{O}}(\mathcal{E}) = \sum_{i \geq 0} \dim_\kappa(\mathrm{gr}_i) \geq \sum_{i=0}^{r} \dim_\kappa(V_i).$$

□

Let $E_{\mathbb{T}}^{\leq r}(k)$ denote the image of the composite $E_0 \xrightarrow{\overline{\mathrm{ev}}_k} W_{\mathbb{T}}(k) \to W_{\mathbb{T}}^{\leq r}(k)$. In the following proof of surjectivity, we start to piece together previous results.

Theorem 5.6. *The map $\pi_0 E(k) \to W_{\mathbb{T}}^{\leq r}(k)$ is surjective for each r.*

Proof. It suffices to show that the induced map $E_{\mathbb{T}}^{\leq r}(k) \to W_{\mathbb{T}}^{\leq r}(k)$ is an isomorphism, and to do that we only need to prove that they have equal lengths as $W(k)$-modules. Since $E^0(B\Sigma_{p^r})/I_{\mathrm{tr}}$ is free of rank $\bar{d}(r)$ by a result from [Str98], using Lemma 5.5 we obtain a lower bound on the length of $E_{\mathbb{T}}^{\leq r}$:

$$\mathrm{len}_{W(k)}\left(E_{\mathbb{T}}^{\leq r}(k)\right) \geq \sum_{j=0}^{r} \bar{d}(j).$$

Specifically, this is obtained by setting $\mathcal{O} = W(k)$, $\mathcal{E} = E_{\mathbb{T}}^{\leq r}(k)$, and $f_i = \overline{P}_{r-i}$ factored through $W_{\mathbb{T}}^{\leq r}(k)$ by Proposition 3.1. The fact that the f_i's are surjective is a consequence of Theorem 4.1, and the condition is the lemma is satisfied because of Lemma 5.2. Now combine with Proposition 3.2, which computes the length of $W_{\mathbb{T}}^{=r}$ as $W(k)$-module to be $\bar{d}(r)$, we arrive at the inequality

$$\sum_{i=0}^{r} \bar{d}(i) \leq \mathrm{len}\left(E_{\mathbb{T}}^{\leq r}(k)\right) \leq \mathrm{len}\left(W_{\mathbb{T}}^{\leq r}(k)\right) = \sum_{i=0}^{r} \mathrm{len}\left(W_{\mathbb{T}}^{=i}(k)\right) = \sum_{i=0}^{r} \bar{d}(i),$$

which implies that $E_{\mathbb{T}}^{\leq r}(k)$ and $W_{\mathbb{T}}^{\leq r}(k)$ have same lengths as $W(k)$-modules. □

References

[AHS04] Matthew Ando, Michael J. Hopkins, and Neil P. Strickland. *The sigma orientation is an H_∞ map*, Amer. J. Math., 126(2):247–334, 2004.

[BF15] Tobias Barthel and Martin Frankland. *Completed power operations for Morava E-theory*, Algebr. Geom. Topol. 15 (2015) 2065–2131.

[BMMS86] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger. *H_∞ ring spectra and their applications*, Lecture Notes in Mathematics, vol. 1176, Springer-Verlag, Berlin, 1986.

[BSY22] Robert Burklund, Tomer M. Schlank, and Allen Yuan. *The Chromatic Nullstellensatz.* July 20, 2022. arxiv: https://arxiv.org/abs/2207.09929

[GM92] J. P. C. Greenlees and J. P. May. *Derived functors of I-adic completion and local homology*, J. Algebra, 149(2), (1992), 438-453.

[HKR00] Michael J. Hopkins, Nicholas J. Kuhn, and Douglas C. Ravenel. *Generalized group characters and complex oriented cohomology theories*, J. Amer. Math. Soc. 13 (2000).

[Law15] Tyler Lawson. *A note on H_∞ structures*, Proceedings of the AMS, Vol. 143, no. 7 (2015)

[Law20] Tyler Lawson. *E_n ring spectra and Dyer-Lashof operations.* 2020. arxiv: https://arxiv.org/abs/2002.03889

[Lur18] Jacob Lurie. *Elliptic Cohomology II: Orientations.* 2018. https://www.math.ias.edu/~lurie/papers/Elliptic-II.pdf

[MNN15] Akhil Mathew, Niko Naumann, and Justin Noel. *On a nilpotence conjecture of J. P. May*, J. Topol., 8(4):917–932, 2015.

[Rez09] Charles Rezk. *The congruence criterion for power operations in Morava E-theory*, Homology Homotopy Appl., 11(2):327–379, 2009.

[Stap] Nathaniel Stapleton. *Lubin-Tate theory, character theory, and power operations*, Handbook of Homotopy Theory, Handbooks in Mathematics Series, CRC Press, Taylor & Francis Group, Boca Raton, FL, 2020, pp. 891-930.

[Str97] Neil P. Strickland. *Finite subgroups of formal groups*, J. Pure Appl. Algebra, 121(2):161–208, 1997.

[Str98] N. P. Strickland. *Morava E-theory of symmetric groups*, Topology, 37(4):757–779, 1998.

[YS] Yunhan (Alex) Sheng. *Power operations and HKR character theory* . https://yunhansheng.github.io/files/power.pdf