Complex Multiplication

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Outline of the talk

1. Number-theoretic background
2. CM of elliptic curves
3. Generalization to abelian Varieties
4. Acknowledgements

Ramification

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- If every prime in K is unramified in L, then L/K is unramified.

The case over $\mathbb{Q}$

- Can we explicitly describe the set of numbers that generates (unramified) abelian extensions of $\mathbb{Q}$?

Theorem 1 (Kronecker-Weber)

Every finite abelian extension of $\mathbb{Q}$ is contained in a cyclotomic extension $\mathbb{Q}(\zeta_N)$ for some $N > 0$.

Theorem 2 (Hermite-Minkowski)

There are no unramified extensions of $\mathbb{Q}$.

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- This is the only known case besides $K = \mathbb{Q}$. The problem is far from being completely resolved.
- Complex Multiplication: this piece of **arithmetic** information will be extracted from studying **geometric** objects, namely, elliptic curves (or more generally, abelian varieties).

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What is an elliptic curve?

By an **elliptic curve** E/K, we understand

- a one-dimensional nonsingular projective variety over K of genus one, together with a special point $O \in E$;
- or more naively, a curve given by so-called **Weierstrass equation**

$$y^2 = x^3 + Ax + B, \quad A, B \in K$$

(N.B. the equation takes this simplified form only if $\mathrm{char}(\overline{K}) \neq 2, 3$.)

Two elliptic curves are isomorphic iff they have the same j-**invariant**:

$$j(E) = \frac{1728(4A)^3}{-16(4A^3 + 27B^2)}.$$

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  is an isogeny.
- Let $\mathrm{End}(E)$ be the ring of isogenies from E to itself, is the map

$$[-] : \mathbb{Z} \to \mathrm{End}(E)$$

  is an isomorphism, or is $\mathrm{End}(E)$ strictly larger than $\mathbb{Z}$?

CM of elliptic curves

Theorem 3

Let $E/\mathbb{C}$ be an elliptic curve. Then either $\mathrm{End}(E) = \mathbb{Z}$ or $\mathrm{End}(E)$ is isomorphic to an order of $\mathbb{Q}(\sqrt{-D})$ for some $D > 0$.

N.B. Let K be a number field. An **order** R of a K is a subring of K that is finitely generated as $\mathbb{Z}$-module and spans K over $\mathbb{Q}$.

For example, $\mathbb{Z}[i]$ and $\{a + 2bi \mid a, b \in \mathbb{Z}\}$ are both orders of $\mathbb{Q}(i)$. The ring of integers is the largest order.

Definition 4

An elliptic curve $E/\mathbb{C}$ has **complex multiplication** (or CM for short) by R if $R = \mathrm{End}(E)$ is an order of an imaginary quadratic field.

Elliptic curves over $\mathbb{C}$

- Two lattice $\Lambda_1, \Lambda_2 \subset \mathbb{C}$ are **homothetic** if $\Lambda_2 = \alpha\Lambda_1$ for some $\alpha \in \mathbb{C}$.

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- (Uniformization) For any $E/\mathbb{C}$, there exists a unique lattice $\Lambda \subset \mathbb{C}$ such that $\mathbb{C}/\Lambda \cong E(\mathbb{C})$ as (complex) Lie groups.
- Conversely, every complex torus arises as an elliptic curve.
- In fact, there is an equivalence of categories between:
  - elliptic curve E over $\mathbb{C}$ with isogenies, and
  - lattices $\Lambda \subset \mathbb{C}$ up to homothety, with

$$\mathrm{Hom}(\Lambda_1, \Lambda_2) = \{\alpha \in \mathbb{C} \mid \alpha\Lambda_1 \subset \Lambda_2\}.$$

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Suppose $E/\mathbb{C} \cong \mathbb{C}/\Lambda$ as Lie groups. Up to homothety replace Λ by $\mathbb{Z} + \tau\mathbb{Z}$ for some $\tau \in \mathbb{C} \setminus \mathbb{R}$. For any $\alpha \in \mathrm{End}(E) \cong \{\alpha \in \mathbb{C} \mid \alpha\Lambda = \Lambda\}$, there exists $m, n, p, q \in \mathbb{Z}$ such that $\alpha = m + n\tau$ and $\alpha\tau = p + q\tau$. Eliminate τ, we get

$$\alpha^2 - (m + q)\alpha + np = 0,$$

so that $\mathrm{End}(E)$ is an integral extension of $\mathbb{Z}$. If $\alpha \notin \mathbb{Z}$, then $n \neq 0$, so eliminating n we get an quadratic equation

$$n\tau^2 + (m - q)\tau - p = 0.$$

Since $\tau \notin \mathbb{R}$, $\mathbb{Q}(\tau)$ is an imaginary quadratic field. □

Construction of class fields

Theorem 5

Let R be an order of an imaginary quadratic field K. Let $E/\mathbb{C}$ be an elliptic curve with CM by R. Then

- *$K(j(E))$ is the maximal unramified extension of K*
- *$K(j(E), x(E_{tors}))$ is the maximal abelian extension of K, where E_{tors} are points of E of finite order, and $x(-)$ is the function taking x-coordinate.*

(N.B. the function $x(-)$ only works if $j(E) \neq 0, 1728$; otherwise we need something more subtle called Weber function.)

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- Moral of the story: j-invariant and coordinate of torsion points generate abelian extensions of $\mathbb{Q}(\sqrt{-D})$ for some $D > 0$.

From numbe theory: idèles

- From number theory: let K be a global field (finite extensions of $\mathbb{Q}$). The completion of K_v at a place (given by an absolute value) v of K is a local field (think about $\mathbb{Q}_p$). Let $\mathcal{O}_v$ be the valuation subring (think about $\mathbb{Z}_p$), the **idèle group** is the topological group

$$\mathbf{A}_K^\times = \left\{ (a_v) \in \prod_v K_v^\times \mid a_v \in \mathcal{O}_v^\times \text{ for all but finitely many } v \right\}.$$

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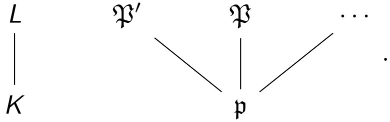
- Packing *local* information in the *global* setting.
- The fractional ideal (x) associated to an idèle $x \in \mathbf{A}_K^\times$ is

$$(x) = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(x_{\mathfrak{p}})},$$

where $(x_{\mathfrak{p}}) = (x)\mathcal{O}_p$.

From number theory: Frobenius substitution

- Let L/K be a finite Galois extension of number fields and $\mathfrak{P}$ a prime lying over an unramified prime $\mathfrak{p}$:



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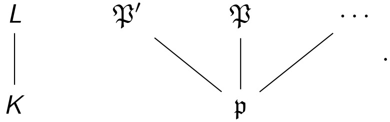

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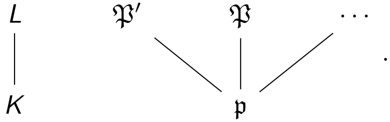

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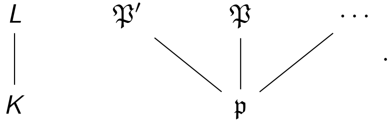

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- If L/K is abelian, then $\sigma_{\mathfrak{P}} = \sigma_{\mathfrak{P}'}$, so we simply write $\sigma_{\mathfrak{p}}$.

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- Class field theory tells us that there is a unique continuous map called the (global) **Artin map**

$$\mathbf{A}_K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

given by $s \mapsto [s, K]$, where if $(s) = \prod_{\mathfrak{p}} \mathfrak{p}^{n_\mathfrak{p}}$ is not divisible by primes that ramify in L, then

$$[s, K]|_L = ((s), L/K) := \prod_{\mathfrak{p}} \sigma_\mathfrak{p}^{n_\mathfrak{p}}$$

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- The Artin map is surjective with $K^\times$ contained in the kernel.

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- Let $s \in \mathbf{A}_K^\times$ be an idèle with $[s, K] = \sigma|_{K^{\mathrm{ab}}}$.
- Let $f : \mathbb{C}/\mathfrak{a} \xrightarrow{\sim} E(\mathbb{C})$ be a complex-analytic isomorphism.

Theorem 6 (The main theorem of CM of elliptic curves)

There exists a unique complex-analytic isomorphism $f' : \mathbb{C}/(s)^{-1}\mathfrak{a} \xrightarrow{\sim} E^\sigma(\mathbb{C})$ *such that the following diagram commutes:*

$$\begin{array}{ccc} K/\mathfrak{a} & \xrightarrow{(s)^{-1}} & K/(s)^{-1}\mathfrak{a} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ E(\mathbb{C}) & \xrightarrow{\sigma} & E^\sigma(\mathbb{C}) \end{array}$$

The associated Hecke character

- A **Hecke character** of a number field K is a continuous map

$$\psi : \mathbf{A}_K^\times \to \mathbb{C}^\times \quad \text{that satisfies } \chi(L^\times) = 1.$$

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- Using the Main Theorem, we can define a Hecke character

$$\psi_{E/L}(s) = \alpha_{L/K}(s)\mathrm{Nm}_{L/K}(s^{-1})_\infty$$

of L/K, where E/L is an elliptic curve with CM by $\mathcal{O}_K$, and $\alpha_{L/K}$ is chosen to make following diagram commutes

$$\begin{array}{ccc} K/\mathfrak{a} & \xrightarrow{\alpha_{L/K}(s)/\mathrm{Nm}_{L/K}s} & K/\mathfrak{a} \\ \sim\downarrow & & \downarrow\sim \\ E^{\mathrm{ab}}(L) & \xrightarrow{[s,L]} & E^{\mathrm{ab}}(L) \end{array}.$$

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- $\psi_{E/L}$ is unramified at $\mathfrak{P}$ of L iff E has good reduction at $\mathfrak{P}$.

L-series of an elliptic curve

The L-series of E/L encodes arithmetic information:

$$L(E/L, s) = \prod_{\mathfrak{P}} L_{\mathfrak{P}}(E/L, q_{\mathfrak{P}}^{-s})^{-1}$$

ranging over primes $\mathfrak{P}$ of L. Each local L-factor is given by

$$L_{\mathfrak{P}}(E/L, T) = 1 - a_{\mathfrak{P}} T + q_{\mathfrak{P}} T^2,$$

where $q_{\mathfrak{P}} = \mathrm{Nm}_{L/\mathbb{Q}} \mathfrak{P}$ and $a_{\mathfrak{P}} = q_{\mathfrak{P}} + 1 - \#\widetilde{E}(\kappa_{\mathfrak{P}})$, $\kappa_{\mathfrak{P}}$ is the residue field of L at $\mathfrak{P}$. In the case when E has bad reduction at $\mathfrak{P}$, we define

$$L_{\mathfrak{P}}(E/L, T) = \begin{cases} 1 - T, & \text{split multiplicative reduction} \\ 1 + T, & \text{non-split multiplicative reduction} \\ 1, & \text{additive reduction} \end{cases}.$$

Hecke L-series

- Let $\psi : \mathbf{A}_L^\times \to \mathbb{C}^\times$ be a Hecke character. Attach to it the L-series

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  - satisfies a functional equation $L(s,\psi) = \epsilon L(N - s, \psi^\vee)$ for some ϵ, N depending on ψ.

A conjecture on the L-series

- The L-series $L(E/L, s)$ of an elliptic curve E/L converges for $\Re s > 3/2$.

Conjecture

The L-series $L(E/L, s)$ has an analytic continuation to the entire complex plane and satisfies a functional equation relating $L(E/L, 2)$ and $L(E/L, 2 - s)$.

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- For E having CM, Deuring and Weil proved that

$$L(E/L, s) = L(s, \psi_{E/L})L(s, \overline{\psi_{E/L}}),$$

so by Hecke's result the conjecture is resolved.

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- For E having CM, Deuring and Weil proved that

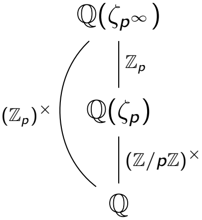
$$L(E/L, s) = L(s, \psi_{E/L})L(s, \overline{\psi_{E/L}}),$$

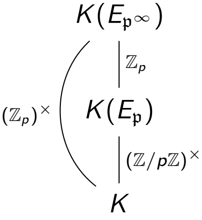
so by Hecke's result the conjecture is resolved.

- Works of Eichler, Shimura, and finally Wiles's modularity theorem resolves the case $E/\mathbb{Q}$.

Iwasawa theory of elliptic curves with CM

- In the classical Iwasawa theory we consider the infinite cyclotomic tower and study the p-adic analogue of Riemann zeta function.





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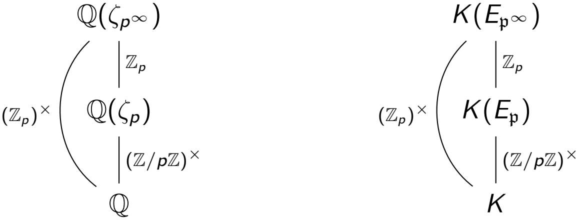

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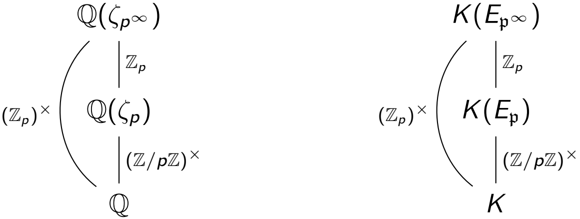

- Substituting $\mathbb{Q}$ by K an imaginary quadratic field, the role of ζ_{p^n} is played by the $\mathfrak{p}^n$-torsion points on E.
- The p-adic $L(E,s)$ tells us the p-part of the Tate-Shafarevich group $Ш(E/\mathbb{Q})$, which is helpful to understanding the BSD conjecture.

Outline of the talk

1. Number-theoretic background
2. CM of elliptic curves
3. Generalization to abelian Varieties
4. Acknowledgements

Facts about abelian varieties

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- Let B be a semisimple K-algebra. By Wedderburn-Artin theorem

$$B = \mathcal{M}_{n_i}(D_i).$$

Let K_i be the center of D_i, define the **reduced degree**

$$[B : K]_{\mathrm{red}} := [B_i : K_i]^{1/2}[K_i : K].$$

It is the degree of the maximal étale K-subalgebra of B.

Abelian variety with CM

Lemma 7

Notation as above, if M is a faithful B-module, then

$$\dim_K M \geq [B : K]_{\mathrm{red}},$$

with equality if and only if B_i are matrix algebras over K_i.

- Fix a uniformization $A \cong \mathbb{C}^g/\Lambda$. Interpret an analytic representation

$$\mathrm{End}_{\mathbb{Q}}(A) \cong \{M \in \mathcal{M}_g(\mathbb{C}) : M\mathbb{Q}\Lambda \subset \mathbb{Q}\Lambda\}.$$

Then $\mathbb{Q}\Lambda$ is a faithful $\mathrm{End}_{\mathbb{Q}}(A)$-module, so that

$$[\mathrm{End}_{\mathbb{Q}}(A) : \mathbb{Q}]_{\mathrm{red}} \leq \dim_{\mathbb{Q}} \mathbb{Q}\Lambda = 2 \dim A.$$

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- We say that $A/\mathbb{C}$ has CM if equality holds.

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- A **CM-algebra** is a finite product of CM-fields.
- By the lemma and the fact that A is semisimple, A has CM if and only if each of its simple factors has CM.

Theorem 8

An abelian variety $A/\mathbb{C}$ has CM if and only if

- *(if A is simple) $\mathrm{End}_{\mathbb{Q}}(A)$ is a CM-field of degree $2\dim A$ over $\mathbb{Q}$;*
- *(if A is isotypic) $\mathrm{End}_{\mathbb{Q}}(A)$ contains a field of degree $2\dim A$ over $\mathbb{Q}$;*
- *$\mathrm{End}_{\mathbb{Q}}(A)$ contains an étale $\mathbb{Q}$-subalgebra of dimension $2\dim A$.*

Moreover, the number field (resp. étale $\mathbb{Q}$-subalgebra) can be chosen to be a CM-field (resp. CM-algebra) invariant under the Rosati involution induced by a polarization of A.

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- Let $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. Let $s \in \mathbf{A}_K^\times$ be an idèle with $[s, K] = \sigma|_{K^{\mathrm{ab}}}$.
- Let $f : \mathbb{C}/\mathfrak{a} \xrightarrow{\sim} E(\mathbb{C})$ be a complex-analytic isomorphism.

Theorem 9 (The main theorem of CM of elliptic curves)

There exists a unique complex-analytic isomorphism $f' : \mathbb{C}/(s)^{-1}\mathfrak{a} \xrightarrow{\sim} E^\sigma(\mathbb{C})$ *such that the following diagram commutes:*

$$\begin{array}{ccc} K/\mathfrak{a} & \xrightarrow{(s)^{-1}} & K/(s)^{-1}\mathfrak{a} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ E(\mathbb{C}) & \xrightarrow{\sigma} & E^\sigma(\mathbb{C}) \end{array}$$

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- Let $\sigma \in \mathrm{Aut}(\mathbb{C}/K^*)$. Let $s \in \mathbf{A}_K^\times$ be an idèle with $[s, K^*] = \sigma|_{(K^*)^{\mathrm{ab}}}$.

Theorem 10 (The main theorem of CM of abelian varieties)

There is a unique isomorphism $\xi' : \mathbb{C}^g / u(\mathrm{Nm}_\Phi(s)^{-1}\mathfrak{a}) \xrightarrow{\sim} A^\sigma$ *such that* A^σ *is of type* $(K, \Phi, \mathrm{Nm}_\Phi(s)^{-1}\mathfrak{a}, \mathrm{Nm}_{K/\mathbb{Q}}((s))\tau)$ *with respect to* ξ'*, and the following diagram commutes:*

$$\begin{array}{ccc} K/\mathfrak{a} & \xrightarrow{\mathrm{Nm}_\Phi(s)^{-1}} & K/\mathrm{Nm}_\Phi(s)^{-1}\mathfrak{a} \\ \downarrow{\scriptstyle \xi \circ u} & & \downarrow{\scriptstyle \xi' \circ u} \\ A & \xrightarrow{\sigma} & A^\sigma \end{array}$$

A little history

- The classical theory of CM was developed by Weber, Fueter, Hasse and Duering before 1950s.
- The main theorem we gave above was restricted over the reflex field K^*. It was due to Shimura, Taniyama, and Weil in the 1950s. It is sufficient for constructing class fields, though.
- The most general case over $\mathbb{Q}$ was proved by Langlands, Tate, and Deligne in the 1980s, also called motivic CM theory.

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Acknowledgements

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Thanks for listening.

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