Elements of complex K-theory

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Abstract

These notes are written as a term paper for a reading course the author did with Prof. Akhil Mathew during Autumn 2022. The focus of this paper is the connection between complex K-theory and cohomology. The motivation is as follows.

In algebraic geometry, one encounters two important objects: vector bundles and algebraic cycles. The first one leads to algebraic K-theory, and the second to motivic cohomology. Algebraic K-theory and motivic cohomology are related via two things: Chern characters and Atiyah-Hirzebruch type spectral sequences. To understand this story (which has been the author's ultimate goal!), one should first understand the more elementary topological analogue, namely how complex K-theory and singular cohomology are related via Chern characters, the Atiyah-Hirzebruch spectral sequence (AHSS), and the Adams operation.

1 Classification of vector bundles and complex K-theory

We assume the readers are familiar with the basic definitions of vector bundles. Unless otherwise stated, all vector bundles will be complex vector bundles. We also assume that all topological spaces are Hausdorff and paracompact. This is not so harsh of a requirement, since all CW complexes are Hausdorff and paracompact. The reason for this requirement is that in constructing the so-called universal bundles, we need to to take inductive limits, and the inductive limit of a sequence of compact spaces is a paracompact space.

We call a vector bundle an n-vector bundle if the fibers have constant dimension n. The first step to the classification of vector bundles is the following observation:

Proposition. *The pullbacks of an n-vector bundle ξ over B along homotopic maps $f, g : A \to B$ are isomorphic bundles over A.*

Then we claim that there exists a space $G_n(\mathbf{C}^\infty)$ and an n-vector bundle γ_n over it, such that any n-vector bundle over a space B is obtained as a pullback of γ_n along to a unique-up-to-homotopy map $B \to G_n(\mathbf{C}^\infty)$. To phrase this functorially:

Theorem. *There is a natural equivalence of functors between* $\mathrm{Vect}_n(-)$ *and* $[-, G_n(\mathbf{C}^\infty)]$.

To construct this mysterious space $G_n(\mathbf{C}^\infty)$, we first start with the Grassmannian $G_n(\mathbf{C}^q)$, the space of n-dimensional subspaces of $\mathbf{C}^q$. We construct a n-vector bundle over $G_n(\mathbf{C}^q)$ is as follows. The total space $E \subset G_n(\mathbf{C}^q) \times \mathbf{C}^q$ consists of pairs (Y, v), where $v \in Y$ are the actual points of the n-dimensional subspace Y, and the projection $E \to G_n(\mathbf{C}^q)$ is simply given by $(Y, v) \mapsto Y$. Take the colimit of $G_n(\mathbf{C}^q)$ and topologize it with weak topology, we get a monstrosity

$$G_n(\mathbf{C}^\infty) := \varinjlim_q G_n(\mathbf{C}^\infty) = \bigcup_q G_n(\mathbf{C}^q).$$

Correspondingly, there is an n-vector bundle over $G_n(\mathbf{C}^\infty)$, and we call this bundle γ_n. In a sense, this n-vector bundle classifies all n vector bundles. Hence it is often called the *universal* n-vector bundle. Due to the fact that every complex vector bundle can be identified with the associated bundle to a $U(n)$-principal bundle, $G_n(\mathbf{C}^\infty)$ is also referred to as the classifying space of $U(n)$, and denoted $BU(n)$.

The isomorphism classes of all vector bundles over a space X forms an abelian monoid $\mathrm{Vect}(X)$. There is a general universal construction of completing an abelian monoid to an abelian group. This produces the K-group $K(X)$, where elements of $K(X)$ are formal differences $[E]-[F]$ of isomorphism classes of vector bundles E and F over X. Tensor product of vector bundles makes $K(X)$ into a ring. The assignment $K(-)$ is contravariant: a map $f : X \to Y$ induces a map $\mathrm{Vect}(Y) \to \mathrm{Vect}(X)$, and thus a ring homomorphism $f^* : K(Y) \to K(X)$.

There are two crucial facts about complex K-theory. The first is Bott periodicity. Denote by $\underline{1}$ the trivial bundle over X. If ξ is a vector bundle, $P(\xi)$ is its associated projective bundle (for definition see Page 4).

Theorem (Bott periodicity)**.** *Let ξ be a line bundle over X. Then $K(P(\xi \oplus 1))$ is a $K(X)$-algebra generated by $[H]$ and subjected to the relation*

$$([H]-[\underline{1}])([\xi][H]-[\underline{1}]) = 0.$$

Note that since $P(\underline{1}\oplus\underline{1}) = X \times S^2$, this gives a weaker statement $K(X)\otimes K(S^2) \simeq K(X\times S^2)$. Therefore, $K(S^{2n+1}) = 0$ and $K(S^{2n}) = \mathbf{Z}$, and thus the name periodicity.

Define higher K-groups $K^n(X) = K(\Sigma^n X)$. Relating to our previous discussion of the classification of vector bundles, we have that

$$K(X) \simeq [X, BU\times \mathbf{Z}],$$

where BU is the colimit taken over $BU(n)$. The second crucial fact about K-theory is that the periodic K-theory functor $K^*(-)$ is a generalized cohomology theory associated to the Ω-spectrum $BU\times\mathbf{Z}$. The deloopings are precisely given by Bott periodicity. Tensor product induces a pairing on the spectrum

$$(BU\times\mathbf{Z})\times(BU\times\mathbf{Z}) \to BU\times\mathbf{Z},$$

and thus a multiplication on $K^*(-)$. Since K-theory is a generalized cohomology theory, what is its relationship with the singular cohomology $H^*(-)$, represented by the Eilenberg-Mac Lane spectrum? This is the question we now go into. The answer is twofold: through Chern characters, and through Atiyah-Hirzebruch spectral sequence. In fact, the latter degenerates rationally to the former.

2 Chern characters and Adams operation

We start with the definition of a characteristic class:

Definition. Let R be a ring. A *characteristic class* c of degree q is a natural assignment $\xi \to c(\xi)$ of each n-vector bundle $\xi = (E, B, p)$ to a cohomology class $c(\xi) \in H^q(B; R)$.

The naturality here means that if ξ' is another n-vector bundle with bundle map $f : \xi \to \xi'$, then $f^*(c(\xi')) = c(\xi)$. To put it more categorically, let $\mathrm{Vect}_n(-) : \mathbf{Top} \to \mathbf{Set}$ be the homotopy invariant functor which sends a topological space X to the set of of isomorphism classes of n-vector bundles over X. Then a characteristic class is a natural transformation $\mathrm{Vect}_n(-) \Rightarrow H^*(-; R)$.

Let Λ_q be the set of all characteristic classes of degree q for n-vector bundles. The addition $c_1 + c_2 : \xi \mapsto c_1(\xi) + c_2(\xi) \in H^q(B; R)$ makes Λ_q into an abelian group. Let $c_1 \in \Lambda_p$ and $c_2 \in \Lambda_q$. Then multiplication $c_2 \cdot c_2 : \xi \mapsto c_1(\xi) \smile c_2(\xi) \in H^{p+q}(B; R)$ makes $\Lambda = \oplus_{q\geq 0}\Lambda_q$ into a graded ring, with identity the assignment $\xi \mapsto 1 \in H^0(B; R)$. Hence Λ_q is referred to as the ring of degree q characteristic classes.

In the classification of vector bundles we see that $\mathrm{Vect}_n(-)$ is represented by $BU(n) = G_n(\mathbf{C}^\infty)$. Therefore characteristic classes are in correspondence with cohomology classes in $H^q(BU(n); R)$. Explicitly, for each $k \in H^q(BU(n); R)$, let $\overline{k} \in \Lambda_q$ be the characteristic class that assigns an n-vector bundle ξ over B to $f^*(\gamma_n) \in H^q(B; R)$, where f is the map $f : \xi \to \gamma_n$ to the universal bundle which is unique up to bundle homotopy. It can be checked that $k \mapsto \overline{k}$ defines a ring isomorphism $H^*(BU(n); R) \to \Lambda$. If the coefficient ring R is $\mathbf{Z}$, then

$$H^*(BU(n); \mathbf{Z}) \cong \mathbf{Z}[c_1, \cdots, c_n],$$

where the c_i's are called Chern classes. For real vector bundles, there is an analogous description $H^*(BO(n); \mathbf{Z}/2\mathbf{Z}) \cong (\mathbf{Z}/2\mathbf{Z})[\omega_1, \cdots, \omega_n]$ where the ω_i's are called Stiefel-Whitney classes.

Without further ado, let us now define Chern classes. From now on we always assume the singular cohomology is with integral coefficient unless otherwise stated.

Definition. There are characteristic classes $c_i : \xi \mapsto H^{2i}(B; \mathbf{Z})$ for $i \geq 0$ for any n-vector bundle ξ over B called *Chern classes* that are uniquely characterized by the following axioms:

(i) $c_0(\xi) = 1$ and $c_i(\xi) = 0$ if $i > n$;

(ii) (Whitney sum formula) $c_i(\xi \oplus \delta) = \sum_{k=0}^{i} c_k(\xi) \smile c_{i-k}(\delta)$;

(iii) $c_1(\gamma_1) = e((\gamma_1)_R)$, the Euler class of the underlying real vector bundle of the universal bundle γ_1 over $\mathbf{CP}^\infty$.

We first need to define what the Euler class is. For that we consider oriented n-vector bundles. Recall that an orientation for a real vector space V is an equivalence of ordered bases, where two bases are equivalent when the transitional matrix between them has positive determinant. Alternatively, a choice of orientation for V is a choice of generator of the infinite cyclic group $H^n(V, V - \{0\})$. For a real n-vector bundle (E, B, p), an orientation is a function assigning each fiber F an orientation as a vector space in the compatible sense: for each $b_0 \in B$ there exists a local trivialization $h : U \times \mathbf{R}^n \to p^{-1}(U)$ such that for each $b \in U$ the map $h|_{b\times\mathbf{R}^n}$ preserves orientation.

It is obvious that in each trivialization U of $b \in B$ there exists some $u \in H^n(p^{-1}(U), p_0^{-1}(U))$ such that for each fiber F in $p^{-1}(U)$ the restriction $u_F = u|_F \in H^n(F, F_0)$ is the orientation of F. The Thom isomorphism theorem below states that such cohomology class u exists globally.

Theorem (Thom)**.** *Let (E, B, p) be an oriented n-vector bundle. Then $H^i(E, E_0) = 0$ for $i < n$, and $H^n(E, E_0)$ contains a unique class u such that the restriction $u|_{(F,F_0)} \in H^n(F, F_0)$ is the orientation generator u_F for any fiber F. Furthermore, the correspondence $x \mapsto x \smile u$ defines an isomorphism $H^k(E) \to H^{k+n}(E, E_0)$ for each k.*

The cohomology class u is referred to as the *fundamental cohomology class* or *Thom class*.

Proof. For a detailed exposition see [6]. The rough idea is that to first prove the statement locally on a trivialization of the bundle. Then argue how two overlapping trivializations fit together via standard Mayer-Vietoris argument. Finally, an induction argument finishes the proof. The same technique is used to prove, for instance, the Leray-Hirsch theorem. □

With this result we now define the Euler class, thereby completing the definition of Chern classes.

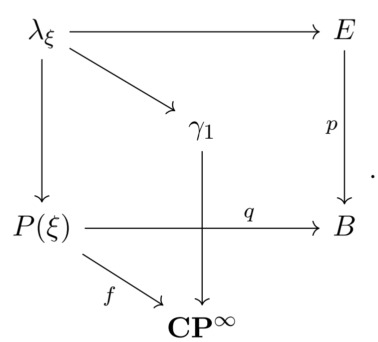
Definition. Let $\xi = (E, B, p)$ be an oriented n-vector bundle with fundamental cohomology class u. Let $i^* : H^n(E, E_0) \to H^n(E)$ be the map induced by the natural inclusion $i : (E, \varnothing) \subset (E, E_0)$. The *Euler class* for ξ is $e(\xi) := p^* i^*(u)$.

The Euler class is natural up to a sign: if $\xi \to \xi'$ is orientation preserving (resp. reversing), then $e(\xi) = f^* e(\xi')$ (resp. $e(\xi) = -f^* e(\xi')$). One also checks that fefw. If ξ is a complex vector bundle, then the underlying real vector bundle ξ_R has a canonical orientation. We write $e(\xi_R)$ for the associated Euler class of ξ.

The question remains how to construct Chern classes. We need the notion of projective bundle.

Definition. Let $\xi = (E, B, p)$ be a n-vector bundle. Let $E_0 \subset E$ be the subspace with the zero section removed. Let $P(\xi)$ be the quotient space of E_0 obtained by identifying points in a fiber that lie in the same one dimensional linear subspace. Let $q : P(\xi) \to B$ be the map that factorizes $p|_{E_0} : E_0 \to B$. The *projective bundle* associated to ξ is the vector bundle $(P(\xi), B, q)$.

If we pullback ξ along $q : P(\xi) \to B$, then we obtain a bundle λ_ξ over $P(\xi)$. It can be checked that λ_ξ is actually a line bundle. Consider the map $f : \lambda_\xi \to \gamma_1$ to the universal bundle:



Denote by a_ξ the pullback of the Euler class $e((\gamma_1)_R)$ along f, which is well-defined since f is unique up to homotopy. The following proposition is a consequence of the Leray-Hirsch Theorem, which gives a fiberwise criteria for the cohomology of the total space to be a free module over the cohomology of the base space.

Proposition. *The classes* $1, a_\xi, \cdots, a_\xi^{n-1}$ *generate* $H^*(P(\xi))$ *as an* $H^*(B)$*-module.*

Thus, we can find cohomology classes $c_i(\xi) \in H^{2i}(B)$ with $c_0(\xi) = 1$, $c_i(\xi) = 0$ for $i > n$, and

$$a_\xi^n = \sum_{j=1}^{n} (-1)^{j+1} c_j(\xi) a_\xi^{n-j}.$$

It can be checked (not easily!) that the classes c_i constructed above are the unique ones that satisfy the axioms of Chern classes. The uniqueness part is a consequence of the splitting principle.

Proposition (The splitting principle)**.** *Given an n-vector bundle $\xi : E \to B$, there exists an associated bundle $f : Fl(\xi) \to B$ called the Flag bundle of ξ such that*

(i) $f^* : H^*(B) \to H^*(Fl(\xi))$ *is injective;*

(ii) $f^*(\xi)$ *splits into a Whitney sum of line bundles.*

With Chern classes at our disposal, we can finally define the object at central of our concern: the Chern character. Chern character nicely relates complex K-theory to singular cohomology, at least if one ignores torsion. Namely, Chern character induces a ring isomorphism

$$\mathrm{ch} : K^*(X) \otimes \mathbf{Q} \xrightarrow{\cong} H^*(X; \mathbf{Q}).$$

The construction of Chern characters will be quite combinatorial in nature. We also acquire Adams operation along the way, so we kill two birds with one stone!

Likewise, we first define Adams operation axiomatically before constructing them explicitly.

Definition. Let X be a compact Hausdorff space. For each $k \geq 0$ there exists a unique ring homomorphism $\psi^k : K(X) \to K(X)$ called the *k-th Adams operation* that satisfy:

(i) (naturality) $\psi^k f^* = f^* \psi^k$ for any $f : X \to Y$;

(ii) $\psi^k(L) = L^k$ if L is an line bundle.

Adams operation is an example of K-theory operations. To construct them, consider the polynomial $Q_k(t_1, \cdots, t_k) = t_1^k + \cdots + t_k^k$ which is invariant under permutations of the variables.

Theorem (Fundamental theorem of symmetric functions)**.** *Every symmetric function $S(t_1, \cdots, t_k)$ can be written uniquely as a polynomial in the elementary symmetric polynomials*

$$\sigma_1 = t_1 + \cdots + t_k, \quad \sigma_2 = \sum_{1 \leq i < j \leq k} t_i t_j, \quad \cdots, \quad \sigma_k = t_1 t_2 \cdots t_k.$$

Therefore, $Q_k(t_1, \cdots, t_k)$ can be expressed as a polynomial in the elementary symmetric polynomials in the variables $t_1, \cdots, t_k$. The expressions $s_k(\sigma_1, \cdots, \sigma_k) = Q_k(t_1, \cdots, t_k)$ are called *Newton polynomials*. For example,

$$s_1 = \sigma_1, \quad s_2 = \sigma_1^2 - 2\sigma_2, \quad \sigma_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3,$$

and so on. Using $\lambda^k(E)$ to denote the exterior power $\wedge^k E$, we define $\psi^k(E) = s_k(\lambda^1(E), \cdots, \lambda^k(E))$. One verifies that it satisfies the axioms in the definition.

Remark. More formally, Adams operation can be defined using the λ-ring structure on K-groups. The λ-ring structure formalizes exterior power. This framework has the advantage that K-theory operations defined thus are readily extended to the algebraic K-theory context. But the present definition involves less definition and is rather elementary.

For an n-vector bundle ξ, let us write $s_k(\xi)$ for $s_k(c_1(\xi), \cdots, c_n(\xi))$. A formal computation shows that for n-vector bundle δ over the same base space,

$$s_k(\xi \oplus \delta) = s_k(\xi) + s_k(\delta), \quad \text{and} \quad s_k(\xi \otimes \delta) = \sum_{i=0}^{k} \binom{k}{i} s_i(\xi) s_{k-i}(\delta).$$

Note that we define $s_k = 0$ for $k > n$.

Definition. Let ξ be an n-vector bundle over B. We define the class $\mathrm{ch}_k(\xi) = s_k(\xi)/k! \in H^{2k}(B)$ for $k \geq 1$ and $\mathrm{ch}_0(\xi) = \dim(\xi)$. The sum $\mathrm{ch}(\xi) = \sum_{k \geq 0} \mathrm{ch}_k(\xi)$ is called the *Chern character* of ξ.

If L is a line bundle, then $\mathrm{ch}(L) = \exp(c_1(L))$. Again, a formal computation gives formula

$$\mathrm{ch}(\xi \oplus \delta) = \mathrm{ch}(\xi) + \mathrm{ch}(\delta), \quad \text{and} \quad \mathrm{ch}(\xi \otimes \delta) = \mathrm{ch}(\xi) \smile \mathrm{ch}(\delta).$$

Therefore, Chern character defines a ring homomorphism $\mathrm{ch} : K^*(B) \to H^*(B)$. In fact, it is an isomorphism if torsion is ignored:

Theorem (Karoubi)**.** *For compact space B, Chern characters induces a ring isomorphism*

$$\mathrm{ch} : K(B) \otimes \mathbf{Q} \longrightarrow \bigoplus_{i \geq 0} \check{H}^{2i}(B; \mathbf{Q}),$$

where $\check{H}$ denotes Čech cohomology.

The Chern character is also an intertwiner of Adams operations ψ^k, in the sense that there is a commutative diagram

$$\begin{array}{ccc} K(B) & \xrightarrow{\mathrm{ch}} & \bigoplus_{i \geq 0} H^{2i}(B; \mathbf{Q}) \\ \downarrow{\scriptstyle \psi^k} & & \downarrow{\scriptstyle \psi^k_H} \\ K(B) & \xrightarrow{\mathrm{ch}} & \bigoplus_{i \geq 0} H^{2i}(B; \mathbf{Q}) \end{array},$$

where ψ^k_H is given by $\psi^k_H(x) = k^r x$ for $x \in H^{2r}(B; \mathbf{Q})$, the eigenvalue of ψ^k on cohomology.

3 Atiyah-Hirzebruch spectral sequence

We first state the (cohomological) Atiyah-Hirzebruch spectral sequence (AHSS), before we sketch its construction. In full generality, AHSS is a spectral sequence that computes a generalized cohomology theory E^* using ordinary cohomology H^* with coefficients in $E^*(\mathrm{pt})$.

Theorem (AHSS)**.** *Let X be a finite CW complex. Let E^* be a generalized cohomology theory. Then there is a spectral sequence with*

$$E_2^{p,q} := H^p(X; E^q(\mathrm{pt})) \Longrightarrow E^{p+q}(X)$$

Every spectral sequence arises from a filtration, and AHSS is no exception. A natural filtration of X given by its CW structure is

$$\mathrm{pt} = X^{-1} \subset X^0 \subset \cdots \subset X^n = X.$$

Let $\widetilde{E}^*$ be the reduced version of E^*. Setting $F_i\widetilde{E}^m(X) = \ker\{\widetilde{E}^m(X) \to \widetilde{E}^m(X^i)\}$ gives a filtration

$$0 = F_n\widetilde{E}^m(X) \subset \cdots \subset F_{-1}\widetilde{E}^m(X) = \widetilde{E}^m(X)$$

of $\widetilde{E}^m$ induced by the CW structure on X. Each entry $E_r^{p,q}$ converges to

$$E_\infty^{p,q} = F_{p-1}\widetilde{E}^{p+q}(X)/F_p\widetilde{E}^{p+q}(X).$$

Remark. According to Adams [2], AHSS was probably first invented by Whitehead. It was first published by Atiyah and Hirzebruch in 1961 with K-theory K^* in place of E^*. Reversing the arrows one gets a homological AHSS. As pointed out in their original paper, AHSS can be generalized to fiber bundles $F \hookrightarrow X \to B$ in the form

$$E_2^{p,q}(X) = H^p(B; E^q(F)) \Longrightarrow E^{p+q}(X)$$

that generalizes also the Serre spectral sequence. In algebraic K-theory, there is a similar spectral sequence that computes algebraic K-groups using motivic cohomology. See Grayson [**?**].

We provide a sketch of the construction of AHSS. Arising from the CW filtration of a space X, each pair (X_p, X_{p-1}) gives a long exact sequence

$$\cdots \to \widetilde{E}^n(X_{p-1}) \xrightarrow{i^*} \widetilde{E}^n(X_p) \xrightarrow{\alpha} E^n(X_p, X_{p-1}) \xrightarrow{j^*} \widetilde{E}^{n-1}(X_{p-1}) \to \cdots$$

which can be further made into an exact couple

$$\begin{array}{ccc} \widetilde{E}^*(X_{p-1}) & \xrightarrow{\quad i^* \quad} & \widetilde{E}^*(X_p) \\ {\scriptstyle \alpha}\nwarrow & & \swarrow{\scriptstyle j^*} \\ & E^*(X_p, X_{p-1}) & \end{array} .$$

The general theory of exact couples allows us to deduce higher pages of the spectral sequence and eventually prove the convergence. The detailed proof is somewhat technical, and we refer to Fomenko-Fuchs [3] for an account.

Remark. Let E be the CW spectrum representing E^*. Then AHSS can be rephrased as

$$E_2^{p,q} = H^p(X; \pi_q(E)) \Longrightarrow E^{p+q}(X).$$

An alternative construction can be given by filtering the Postnikov tower of the spectrum E, where each layer of the tower is has the shifted Eilenberg-Mac Lane spectrum $\Sigma^n H\pi_n(E)$ as homotopy fiber. Smashing with X, the Postnikov system gives a long exact sequence of homotopy groups that assembles to an exact couples.

Complex K-theory is a generalized cohomology theory, and therefore AHSS applies. Due to Bott periodicity, all odd rows in the K-theory AHSS are zero since $K^i(\text{pt}) = 0$ for i odd. Consequently, differentials d_i are also zero for i even, and thus all information is contained in a single row of the original spectral sequence.

Corollary (AHSS for Complex K-theory)**.** *Let X be a finite CW complex. Then there is a spectral sequence with*

$$E_2^p = H^p(X; \mathbf{Z}) \Longrightarrow K^p(X).$$

The differential $d_r^p : E_r^p \to E_r^{p+r}$ is zero if r is even, and each E_r^p converges to

$$E_\infty^p = \frac{\ker\{K^p(X) \to K^p(X^{p-2})\}}{\ker\{K^p(X) \to K^p(X^p)\}}.$$

Let us now compute some honest-to-god examples using AHSS.

Example (K-theory of $\mathbf{CP}^n$)**.** The E_2 page of the spectral sequence looks like

$$\left[\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & \cdots & 2n & 2n+1 & 2n+2 & \cdots \\ \hline \mathbf{Z} & 0 & \mathbf{Z} & 0 & \cdots & \mathbf{Z} & 0 & 0 & \cdots \end{array}\right]$$

All differentials $d_r^p : \mathbf{Z} \to \mathbf{Z}$ are zero since r is even. Thus the $E_2 = E_\infty$. Since $\mathbf{Z}$ is projective, there are no extension problems. We conclude that $K^0(\mathbf{CP}^n) = \mathbf{Z}^{n+1}$ and $K^1(\mathbf{CP}^n) = 0$. Note that it is possible to do this computation by induction on n without using spectral sequence. However, the present computation is so much simpler.

Can we further deduce the ring structure of $K^*(\mathbf{CP}^n)$? The idea is that we know the ring structure of $H^*(\mathbf{CP}^n; \mathbf{Z})$, and Chern characters give us a path from H^* to K^*.

Proposition. *Let ζ be the tautological bundle $\mathcal{O}_{\mathbf{P}^1}(-1)$ over $\mathbf{CP}^1$, i.e., the Hopf bundle over S^2. Let ξ be the line bundle over $\mathbf{CP}^n$ obtained by pullback of ζ. Let $\underline{1}$ be the trivial line bundle over $\mathbf{CP}^n$. Let $\gamma = \xi - \underline{1}$. Then as a ring,*

$$K^*(\mathbf{CP}^n) = \mathbf{Z}[\gamma]/\gamma^{n+1}.$$

Proof. Since ξ is a line bundle, $\mathrm{ch}(\gamma) = \mathrm{ch}(\xi - \underline{1}) = e^x - 1$, where $x = c_1(\gamma)$. We have

$$\mathrm{ch}(\gamma^k) = \mathrm{ch}(\gamma)^k = (e^x - 1)^k = (1 + x + x^2/2 + \cdots)^k = x^k + kx^{k+1}/2 + \cdots$$

for $1 \le k \le n$ and $\mathrm{ch}(\gamma^k) = 0$ for $k > n$. By the ring isomorphism induced by Chern character, $1, \gamma, \cdots, \gamma^n$ are linearly independent in $K^*(\mathbf{CP}^n)$ and generates $K^*(\mathbf{CP}^n)$ over $\mathbf{Q}$. It suffices to show that they generate $K^*(\mathbf{CP}^n)$ over $\mathbf{Z}$. We argue this by induction on n. Obviously the case $n = 1$ holds. Now suppose it also holds for $\mathbf{CP}^{n-1}$. Then for

$$\alpha = a_0 + a_1\gamma + \cdots + a_n\gamma^n \in K^*(\mathbf{CP}^n),$$

restriction to $\mathbf{CP}^{n-1}$ shows that $a_0, \cdots, a_{n-1} \in \mathbf{Z}$. To show that $a_n \in \mathbf{Z}$, consider the portion of the long exact sequence in K-groups

$$\cdots \longrightarrow \widetilde{K}(S^{2n}) = K(\mathbf{CP}^n, \mathbf{CP}^{n-1}) \xrightarrow{j^*} K(\mathbf{CP}^n) \xrightarrow{i^*} K(\mathbf{CP}^{n-1}) \longrightarrow \cdots .$$

Exactness at $K(\mathbf{CP}^n)$ implies $a_n\gamma^n \in \ker i^* = \mathrm{Im} j^*$. But the Chern character on $\widetilde{K}(S^{2n})$ has image $\mathbf{Z}$, so it must be the case that $a_n \in \mathbf{Z}$. □

Example (K-theory of $\mathbf{RP}^2 \times \mathbf{RP}^4$)**.** This is a trickier example. For product spaces, we cite a Künneth formula result from Atiyah [1], which says that for finite CW complexes X and Y, there is a $\mathbf{Z}/2\mathbf{Z}$-graded exact sequence

$$0 \longrightarrow K^*(X) \otimes K^*(Y) \xrightarrow{\alpha} K^*(X \times Y) \xrightarrow{\beta} \mathrm{Tor}(K^*(X), K^*(Y)) \longrightarrow 0 ,$$

where $\deg \alpha = 0$ and $\deg \beta = 1$. It is worth mentioning that this fails for real K-theory. Now to be able to apply this to our present case, we need to compute the K-theory of $\mathbf{RP}^2$ and $\mathbf{RP}^4$. For $\mathbf{RP}^4$, the E_2 page looks as follows

$$\left[\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ \hline \mathbf{Z} & 0 & \mathbf{Z}/2\mathbf{Z} & 0 & \mathbf{Z}/2\mathbf{Z} & 0 & 0 & \cdots \end{array}\right].$$

Immediately we know that $K^1(\mathbf{RP}^4) = 0$. For $K^0(\mathbf{RP}^4)$, it is not obvious how the successive quotients $\mathbf{Z}$, $\mathbf{Z}/2\mathbf{Z}$, and $\mathbf{Z}/2\mathbf{Z}$ fit together. It turns out that $K^0(\mathbf{RP}^4) = \mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z}$. Similarly, it can be shown that $K^1(\mathbf{RP}^2) = 0$ and $K^0(\mathbf{RP}^2) = \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. Now the Künneth formula applies, and we get

$$\begin{array}{c} 0 \\ \downarrow \\ K^0(\mathbf{RP}^2) \otimes K^0(\mathbf{RP}^4) \oplus K^1(\mathbf{RP}^2) \otimes K^1(RP^4) \\ \downarrow \\ K^0(\mathbf{RP}^2 \times \mathbf{RP}^4) \\ \downarrow \\ \mathrm{Tor}(K^1(\mathbf{RP}^2), K^0(\mathbf{RP}^4)) \oplus \mathrm{Tor}(K^0(\mathbf{RP}^2), K^1(\mathbf{RP}^4)) \\ \downarrow \\ 0 \end{array}$$

In a spectral sequence, it is often quite difficult to determine the differentials explicitly. Let us illustrate the following nontrivial observation regarding K-theory AHSS:

Proposition. *The first possibly nonzero differential*

$$d_3 : H^n(X; \mathbf{Z}) \to H^{n+3}(X; \mathbf{Z})$$

on the E_3 page is given by $\widetilde{Sq}^3$, where $\widetilde{Sq}$ recovers the Steenrod square Sq^3 after reduction mod 2.

Proof. Since $K^1(\mathrm{pt}) = 0$, all differentials d_2 on the E_2 page vanishes for dimensional reasons. Hence the first possibly nonzero differential is d_3, going from E_2^n to E_2^{n+3}. We claim that d_3 is a stable cohomology operation.

Since cohomology operations $\theta : H^n(-, G) \to H^m(-, G')$ are in bijective correspondence with $[K(G, n), K(G', m)]$, we compute that

$$[K(\mathbf{Z}, n), K(\mathbf{Z}, n+3)] \cong H^{n+3}(K(\mathbf{Z}, n); \mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z},$$

where the cohomology of Eilenberg-Mac Lane space can be computed using the Serre spectral sequence. This implies that there exists a unique nonzero stable integral cohomology operation that raises degree by 3. One construction of such operation is given by

$$\widetilde{Sq}^3 : \; H^n(X; \mathbf{Z}) \xrightarrow{\alpha} H^n(X; \mathbf{Z}/2\mathbf{Z}) \xrightarrow{Sq^2} H^{n+2}(X; \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\beta} H^{n+3}(X; \mathbf{Z}) \; ,$$

where α is reduction mod 2 and β is the Bockstein homomorphism. To see that $d_3 = \widetilde{Sq}^3$, it suffices to demonstrate a space X for which there exists a nonzero d_3. Take $X = \mathbf{RP}^2 \times \mathbf{RP}^4$ as in the previous example. Then the map

$$d_3^3 : \mathbf{Z}/2\mathbf{Z} \cong H^3(X; \mathbf{Z}) \to H^6(X; \mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$$

is an isomorphism. This concludes the proof. □

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