2 Brownian motion

2.1 Lévy's construction of BM

Consider the 1-d case, let B_t be the position (of a particle) at time t.

(i) (start at origin) $B_0 = 0$, for convenience purposes;

(ii) (independent increment) $B_t - B_s$ is independent of the σ-algebra generated by $\{B_r : r \leq s\}$, for $s < t$;

(iii) (identically distributed increments) $B_t - B_s$ has the same distribution as $B_{t-s} - B_0$, for $s < t$.

(iv) (continuity w.p.1) the function $t \mapsto B_t$ is continuous w.p.1

A process satisfying (i)-(iii) is a **Lévy process**. For instance, a Poisson process is a Lévy process but does not satisfy (iv)

Proposition 2.1. *If a process satisfies (i)-(iv), then there exists some $\mu \in \mathbb{R}$ and $\sigma^2 \geq 0$ s.t. $B_t - B_s \sim N(\mu(t-s), \sigma^2(t-s))$.*

A **Brownian motion** (or **Wiener Process**) with drift μ and variance parameter σ^2 starting at the origin is a stochastic process (a collection of random variables indexed by time t) $\{B_t : t \geq 0\}$ satisfying (i), (ii), (iv), and $B_t - B_s \sim N(\mu(t-s), \sigma^2(t-s))$ for $s < t$.

Proposition 2.2. *If B_t is a standard Brownian motion, then $Y_t = \sigma B_t + t\mu$ is a Brownian motion with drift μ and variance parameter σ^2.*

Does Brownian motion exist? We give a concrete construction.

Lemma 2.3. *Let X and Y be independent normal variables with zero mean and σ^2 variance, then $X+Y$ and $X-Y$ are independent normal variables with zero mean and $2\sigma^2$ variance.*

Construction. By Proposition 1.2, it suffices to construct a standard BM. It also suffices to consider only the interval $[0,1]$ by definition.

Consider the dyadic integers $D_n = \{j2^{-n} : j = 0, 1, \cdots, 2^n\}$ and $D = \cup_{n\geq 0} D_n$. We first construct B_t for $t \in D$ and prove that it is uniformly continuous in t w.p.1. Since uniformly continuous functions defined on a dense set has a unique extension, we simply define $B_t = \lim_{n\to\infty} B_{t_n}$ for $t_n \to t$ and $t_n \in D$.

Let $\{N_q : q \in D\}$ be a collection of independent standard normal random variables, then we define

$$\begin{cases} B_1 = N_1 \\ B_{1/2} = B_1/2 + N_{1/2}/2 \\ B_{1/4} = B_{1/2}/2 + N_{1/4}/2^{3/2} \\ B_{3/4} = 1 - B_{1/4} \\ \cdots \\ B_d = B_{d+1/2^n}/2 + N_d/2^{(n+1)/2} \text{ for } d \in D_n \setminus D_{n-1} \end{cases}.$$

By Lemma 1.3, $\{B_{1/2^n}, B_{2/2^n} - B_{1/2^n}, \cdots, B_1 - B_{2^n-1/2^n}\}$ are independent $N(0, 1/2^n)$ variables. We are left to show that $q \mapsto B_1$ is uniformly continuous w.p.1.

It suffices to show that w.p.1, for any $\alpha < 1/2$ and

$$K_n = \sup\{|B_q - B_s| : q, s \in D \text{ and } |q-s| \le 1/2^n\},$$

$\lim_{n\to\infty} 2^{\alpha n} K_n = 0$ (in particular $K_n \to 0$). Let

$$Y(j,n) = \sup\{\left|B_1 - B_{(j-1)/2^n}\right| : q \in D \cap [(j-1)/2^n, j/2^n]\}$$

and $J_n = \max\{Y(j,n) : j = 1, \cdots, 2^n\}$, then $J_n \le 3K_n$, and it suffices to substitute K_n for J_n. Observe that $Y(1,n), \cdots, Y(2^n, n)$ are i.i.d., and $Y(1,n)$ has the same distribution as $Y(1,0)/2^{n/2}$ (the reason for $2^{n/2}$ instead of 2^n is because while the variance is halved, the distribution is scaled by a factor of $\sqrt{2}$). Hence,

$$\begin{aligned}
\mathbb{P}\{J_n \ge \epsilon_n\} &= \mathbb{P}\{\max Y(j,n) \ge \epsilon_n\} \\
&\le \sum_{j=1}^{2^n} \mathbb{P}\{Y(j,n) \ge \epsilon_n\} \\
&= 2^n \mathbb{P}\{Y(1,n) \ge \epsilon_n\} \\
&= 2^n \mathbb{P}\{Y(1,0) \ge 2^{n/2}\epsilon_n\}.
\end{aligned}$$

By Borel-Cantelli lemma, if we can find ϵ_n such that $\sum_{n\ge1} \mathbb{P}\{J_n \ge \epsilon_n\} < \infty$, then $J_n \le \epsilon_n$ for large enough n. We need a lemma:

Lemma 2.4 (reflection principle (for dyadics))**.** *For any $a > 0$, $\mathbb{P}\{Y(1,0) > a\} \le 4\mathbb{P}\{B_1 \ge a\}$.*

Proof. Since $\mathbb{P}\{Y(1,0) > a\} \le 2\mathbb{P}\{\sup_{q\in D} B_q \ge a\}$, it suffices to show that $\mathbb{P}\{\max_{j=1,\cdots,2^n} B_{j/2^n} > a\} \le 2\mathbb{P}\{B_1 \ge a\}$ for any n. Fix n and let

$$A_j = \{B_{j/2^n} > a \text{ and } B_{i/2^n} \le a \text{ for } i = 1, \cdots, j-1\}$$

be the first time that $B_{j/2^n}$ is larger than a, then $\{\max_{j=1,\cdots,2^n} B_{j/2^n} > a\} = \sqcup_{k=1}^{2^n} A_k$. Also observe that $\mathbb{P}(B_1 > a | A_k) \ge 1/2$. Hence

$$\mathbb{P}\{\max_{j=1,\cdots,2^n} B_{j/2^n} > a\}/2 = \sum_{k=1}^{2^n} \mathbb{P}(A_k)/2 \le \sum_{k=1}^{2^n} \mathbb{P}\{B_1 > a; A_k\} = \mathbb{P}\{B_1 > a\}.$$

□

For large enough a, we have the following estimate:

$$\mathbb{P}\{Y(1,0) > a\} \le 4\mathbb{P}\{B_1 \ge a\} = 4\int_a^\infty e^{-x^2/2}dx/\sqrt{2\pi} \le 4\int_a^\infty e^{-xa/2}dx/\sqrt{2\pi} \le Ce^{-a^2/2}/a \le e^{-a^2/2}$$

where C is some constant. Hence we have

$$\mathbb{P}\{J_n \ge \epsilon_n 2^{-n/2}\} = 2^n \mathbb{P}\{Y(1,0) \ge \epsilon_n\} \le 2^n e^{-\epsilon_n^2/2} = e^{n(\log 2 - \beta^2/2)}$$

if we choose $\epsilon_n = \beta\sqrt{n}$. Hence if we choose $\beta^2 > 2\log 2$, then $\sum_{n\ge1} \mathbb{P}\{J_n \ge \beta\sqrt{n}2^{-n/2}\} < \infty$ as desired, and by Borel-Cantelli lemma $J_n \le \beta\sqrt{n}2^{-n/2}$. Hence $\lim_{n\to\infty} 2^{\alpha n} J_n = 0$ for $\alpha < 1/2$.

Finally, one can check that $B_t = \lim_{n\to\infty} B_{t_n}$ for $t_n \to t$ and $t_n \in D$ does define a Brownian motion.

Remark. We need out probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to be large enough to have a countable collection of independent standard normal random variables. Luckily, $[0,1]$ with Borel σ-field and Lebesgue measure suffices (HW1).

2.2 Properties of BM

Proposition 2.5. *W.p.1, Brownian motion $\{B_t\}_{0\le t\le 1}$ is not differentiable at t for all $t\in[0,1]$.*

Proof. Suppose event $E_M = \{\exists t\in[0,1]$ such that $|B'_t|\le M\}$ holds, then by continuity there exists ϵ s.t. $|s-t|<\epsilon$ implies $|B_t - B_s| < 2M\epsilon$. Then if $|s-t|, |s'-t| < \epsilon$, then $|B_s - B_{s'}| \le 4M\epsilon$. For large enough n, this means that there exists $k\le n$ s.t.

$$A_{M,n}\begin{cases}\left|B_{(k+1)/n} - B_{k/n}\right| \le 4M/n\\ \left|B_{(k+2)/n} - B_{(k+1)/n}\right| \le 4M/n\\ \left|B_{(k+3)/n} - B_{(k+2)/n}\right| \le 4M/n\end{cases}.$$

To show that the countable union $\mathbb{P}(\cup_{M\ge1}E_M)=0$, it suffices to show that $\mathbb{P}(E_M)=0$. Observe that $\mathbb{P}(E_M)\le\mathbb{P}\{\exists k\le n : A_{M,n}\}$, so it suffices to show that $\lim_{n\to\infty}\mathbb{P}(A_{M,n})=0$. Exactly,

$$\begin{aligned}
\mathbb{P}(A_{M,n}) &= \mathbb{P}\left\{\min_{0\le k\le n-1}\left[\max\begin{cases}\left|B_{(k+1)/n} - B_{k/n}\right|,\\ \left|B_{(k+2)/n} - B_{(k+1)/n}\right|,\\ \left|B_{(k+3)/n} - B_{(k+2)/n}\right|\end{cases}\right\}\le 4M/n\right]\right\}\\
&\le \sum_{k=0}^{n-1}\mathbb{P}\left\{\max\begin{cases}\left|B_{(k+1)/n} - B_{k/n}\right|,\\ \left|B_{(k+2)/n} - B_{(k+1)/n}\right|,\\ \left|B_{(k+3)/n} - B_{(k+2)/n}\right|\end{cases}\right\}\le 4M/n\right\}\\
&= n\mathbb{P}\left\{\max\{\left|B_{1/n}\right|, \left|B_{2/n}-B_{1/n}\right|, \left|B_{3/n}-B_{2/n}\right|\}\le 4M/n\right\}\\
&= n[\mathbb{P}\{\left|B_{1/n}\le 4M/n\right|\}]^3\\
&= n[\mathbb{P}\{\left|\sqrt{n}B_{1/n}\le 4M/\sqrt{n}\right|\}]^3\\
&= n\left[\int_0^{4M/\sqrt{n}} e^{-x^2/2}dx/\sqrt{2\pi}\right]^3 \le CM^3 n/(\sqrt{n})^3 \to 0
\end{aligned}$$

as $n\to\infty$. □

Remark. Proposition 1.5 is stronger than: for any $t\in[0,1]$, B_t is not differentiable at t w.p.1.

The **natural filtration** for Brownian motion is $\mathcal{F}_t = \sigma\{B_s : s\le t\}$. We say a process B_t is adapted to a filtration $\{\mathcal{F}_t : t\ge 0\}$ if for B_t is $\mathcal{F}_t$-measurable for each t. Recall that a **martingale** w.r.t. $\mathcal{F}_t$ is an $\mathcal{F}_t$-adapted process such that $\mathbf{E}(|M_t|)<\infty$ for any t and $\mathbf{E}(M_t|\mathcal{F}_s)=M_s$ for any $s<t$. A martingale is continuous (not merely continuous-time!) if $t\mapsto M_t$ is a continuous function w.p.1. Observe that the *standard* Brownian motion is a continuous martingale:

$$\mathbf{E}(B_t|\mathcal{F}_s) = \mathbf{E}(B_s + (B_t - B_s)|\mathcal{F}_s) = \mathbf{E}(B_s|\mathcal{F}_s) + \mathbf{E}((B_t-B_s)|\mathcal{F}_s) = B_s + \mathbf{E}(B_t - B_s) = B_s.$$

A random variables $T:\Omega\to[0,\infty]$ is a stopping time w.r.t. $\{\mathcal{F}_t\}$ if $\{T\le t\}\in\mathcal{F}_t$ for all $t\ge 0$. Also define $\mathcal{F}_T = \sigma\{A : A\cap\{T\le t\}\in\mathcal{F}_t$ for all $t\}$.

Proposition 2.6 (reflection principle)**.** *Let B_t be a Brownian motion with drift $\mu=0$, then for any $a>0$, $\mathbb{P}\{\max_{0\le s\le t}B_s\ge a\} = 2\mathbb{P}\{B_t\ge a\}$.*

Proof. Let $T_a = \min\{s : B_s\ge a\} = \min\{s : B_s = a\}$, then $\mathbb{P}\{\max_{0\ge s\ge t}B_s\ge a\} = \mathbb{P}\{T_a\le t\} = \mathbb{P}\{T_a<t\}$. Also $\mathbb{P}\{B_t\ge a|t>T_a\}=1/2$, hence

$$\mathbb{P}\{B_t\ge a\} = \mathbb{P}\{T_a<t\}\mathbb{P}\{B_t\ge a|T_a<t\} = \mathbb{P}\{T_a<t\}/2.$$

□

Proposition 2.7 (weak Markov property)**.** *Let B_t be a Brownian motion and $Y_s = B_{s+t} - B_t$, then $\{Y_s : s \geq 0\}$ is a Brownian motion with the same μ and σ^2 as B_t, and is independent of $\mathcal{F}_t = \sigma\{B_s : s \geq t\}$.*

Proposition 2.8 (strong Markov property)**.** *Let B_t be a $BM(\mu, \sigma^2)$ and τ a stopping time w.r.t the filtration $\{\sigma(B_t)\}$. Suppose that $\mathbb{P}\{\tau < \infty\} = 1$. Then $Y_t = B_{t+\tau} - B_\tau$ is a $BM(\mu, \sigma^2)$ independent of $\sigma\{B_t : t \in [0, \tau]\}$.*

Proposition 2.9. *Let B_t be a Brownian motion and T be a stopping time all w.r.t. $\{\mathcal{F}_t\}$. Suppose that $\mathbb{P}\{T \leq \infty\} = 1$. Let $Y_t = B_{t+T} - B_T$ for $0 \leq t < \infty$. Then Y_t is a Brownian motion with the same μ and σ^2, and is independent of $\mathcal{F}_t$.*

Example. We wish to find $q(r, s) = \mathbb{P}\{B_t = 0 \text{ for some } t \in [r, s]\}$ where B_t is a standard Brownian motion.

Lemma 2.10. *Let $A_1, A_2, \cdots$ be events and $\mathcal{F}_n = \sigma\{A_1, \cdots, A_n\}$. Suppose that there exists $b_n > 0$ such that $\sum_{n\geq 1} b_n = \infty$ and $\mathbb{P}(A_{n+1}|\mathcal{F}_n) \geq b_n$ for any n. Then A_n occurs infinitely often.*

Remark. This lemma generalizes the Borel-Cantelli lemma: if $A_1, A_2, \cdots$ are independent events and $\sum_{n\geq 1} \mathbb{P}(A_n) = \infty$ then A_n occurs infinitely often.

Proposition 2.11 (law of iterated logarithm)**.** *Let B_t be a standard BM. Then w.p.1,*

$$\limsup_{t\to\infty} B_t/\sqrt{2t \log\log t} = 1.$$

Proof. We first show that $\limsup_{t\to\infty} B_t/\sqrt{t \log\log t} \leq \sqrt{2}$ w.p.1. It suffices to show that for any $\epsilon > 0$ w.p.1 for all t sufficiently large, $B_t \leq \sqrt{2(1+\epsilon)t \log\log t}$. We use a trick called geometric scaling. Let $\rho > 1$ and $A_n^{\epsilon,\rho} = \{B_t \geq \sqrt{2(1+\epsilon)t \log\log t} \text{ for some } \rho^{n-1} \leq t \leq \rho^n\}$. By Borel-Cantelli lemma, it suffices to show that for any $\epsilon > 0$ there exists $\rho > 1$ such that $\sum_{n\geq 1} \mathbb{P}(A_n^{\epsilon,\rho}) < \infty$. By the reflection principle,

$$\begin{aligned}
\mathbb{P}(A_n^{\epsilon,\rho}) &\leq \mathbb{P}\left\{\sup_{0\leq t\leq \rho^n} B_t \geq \sqrt{2(1+\epsilon)\rho^{n-1}\log\log \rho^{n-1}}\right\} \\
&\leq 2\mathbb{P}\left\{B_{\rho^n} \geq \sqrt{2(1+\epsilon)\rho^{n-1}\log\log \rho^{n-1}}\right\} \\
&= 2\mathbb{P}\left\{B_{\rho^n}/\sqrt{\rho^n} \geq \sqrt{2(1+\epsilon)(\log(n-1) + \log\log\rho)/\rho}\right\} = C_\rho (n-1)^{-(1+\epsilon)/\rho}
\end{aligned}$$

for some constant C_ρ depending on ρ. It suffices to take ρ so that $(1+\epsilon)/\rho > 1$ to make it summable.

For the other side of the inequality, we will show that for any $\epsilon > 0$ there exists $\rho > 1$ such that $\limsup_{n\to\infty} B_{q^n}/\sqrt{2(1-\epsilon)\rho^n \log\log \rho^n} \geq 1$ w.p.1. Let $A_n = \{|B_{q^n}| \geq \sqrt{2(1-\epsilon)\rho^n \log\log\rho^n}\}$. Since $\mathbb{P}(A_n|\mathcal{F}_{n-1}) \geq \mathbb{P}\{B_{\rho^n} - B_{\rho^{n-1}} \geq \sqrt{2(1-\epsilon)\rho^n \log\log\rho^n}\}/2 = \mathbb{P}(\tilde{A}_n)/2$, where $\mathcal{F}_{n-1} = \sigma\{A_1, \cdots, A_{n-1}\}$, by the previous lemma it suffices to show that $\sum_{n\geq 1} \mathbb{P}(\tilde{A}_n) = \infty$. Observe that

$$\begin{aligned}
\mathbb{P}(\tilde{A}_n) &= \mathbb{P}\left\{(B_{\rho^n} - B_{\rho^{n-1}})/\sqrt{\rho^n - \rho^{n-1}} \geq \sqrt{2(1-\epsilon)\rho(\log n + \log\log\rho)/(\rho-1)}\right\} \\
&\geq C_{\epsilon,\rho} \exp\{-(1-\epsilon)\rho(\log n + \log\log\rho)/(\rho-1)\}/\sqrt{2(1-\epsilon)\rho(\log n + \log\log\rho)/(\rho-1)} \\
&\geq C'_{\epsilon,\rho} n^{-(1-\epsilon)\rho/(\rho-1)}/\sqrt{\log n}.
\end{aligned}$$

Again, take ρ_ϵ large enough so that $(1-\epsilon)\rho/(\rho-1) < 1$ to make unsummable. □

Remark. The common strategy in proving these statements is to use Borel-Cantelli lemma first. We often need some scaling and to investigate the difference of Brownian paths $(B_{\rho^n} - B_{\rho^{n-1}})$ Then to bound the probability of individual events, use tools like reflection principle and take integration of the Gaussian variable.

Let B_t be a standard BM, the **zero set** of it is $Z_t = \{s \in [0,t] : B_s = 0\}$. We write $Z = Z_\infty$. A point $s \in Z_t$ is **right-isolated** if there exists $\epsilon > 0$ such that $r \notin Z$ for $r \in (s, s+\epsilon)$. A point in $\mathbb{Z}_t$ is **isolated** if it is both left- and right-isolated.

Proposition 2.12. *We make the following observations:*

(i) $0 \in Z_t$*;*

(ii) Z_t *is closed for any* t*;*

(iii) 0 *is not right isolated;*

(iv) Z *is unbounded;*

(v) Z *has no isolated points.*

Proof. (i)-(iii) is immediate. To prove (iv), we use intermediate value theorem together with, say, the law of iterated logarithm. Alternatively we can use the inverse ....

For (v), □

2.3 Quadratic variation of BM

2.4 Dimension of BM

2.5 BM in $\mathbb{R}^d$

Definition. Let $B_t^1, \cdots, B_t^d$ be a sequence of independent (1-dimensional) standard BMs, then $B_t = (B_t^1, \cdots, B_t^d)$ is a standard d-dimensional BM.

Lemma 2.13. *For standard d-dimensional BM* B_t*, the following holds:*

(i) $B_t = 0$*;*

(ii) if $s < t$*, then* $B_t - B_s$ *is independent of* $\mathcal{F}_s$*;*

(iii) $(B_t - B_s) \sim N(0, (t-s)I)$ *where* I *is the identity matrix, and the density is given by*

$$f(x_1, \cdots, x_n) = \prod_{j=1}^{d} (2\pi(t-s))^{-1/2} \exp\left\{\frac{-x_j^2}{2(t-s)}\right\} = (2\pi(t-s))^{-d/2} \exp\left\{\frac{-|x^2|}{2(t-s)}\right\};$$

if B_t *has drift* μ *and covariance matrix* Γ*, then* $(B_t - B_s) \sim N((t-s)\mu, (t-s)\Gamma)$ *instead;*

(iv) w.p.1 $t \mapsto B_t$ *is continuous;*

(v) $Y_t = \mu t + AB_t$ *is a BM with drift* μ *and covariance matrix* AA^T *for* $\mu \in \mathbb{R}^d$ *and* $A \in M_d(\mathbb{R})$*.*

2.6 BM as martingales

Let M_t be a continuous martingale and τ a stopping time, both adapted to the filtration $\{\mathcal{F}_t\}$. Let $\hat{M}_t = M_{t\wedge\tau}$ and suppose that $\mathbb{P}\{\tau < \infty\} = 1$. Then $\hat{M}_\infty = M_\tau$, and $\mathbf{E}(M_0) = \mathbf{E}(\hat{M})$ for every t. The *optional stopping theorem* says that if $\lim_{t\to\infty} \mathbf{E}(\hat{M}_t) = \mathbf{E}(\hat{M}_\infty)$, then $\mathbf{E}(M_\tau) = \mathbf{E}(M_0)$.

Example (gambler's ruin for BM)**.** Let B_t be a standard BM and $\tau = \inf\{t : B_t = a \text{ or } B_t = -b\}$ be the stopping time. Then $\mathbb{P}\{T < \infty\} = 1$ by the recurrence property of BM, and $\mathbf{E}(B_0) = 0$. Since $\mathbf{E}(B_\tau) = \lim_{t\to\infty} \mathbf{E}(B_{t\wedge\tau})$ by the dominating convergence theorem, we get $0 = \mathbf{E}(B_\tau) = a\mathbb{P}\{B_\tau = a\} - b\mathbb{P}\{B_\tau = -b\}$ which solves to $\mathbb{P}\{B_\tau = a\} = b/(a+b)$ and $\mathbb{P}\{B_\tau = -b\} = a/(a+b)$.

2.7 Harmonic functions in $\mathbb{R}^d$

Let D be a connected open subset of $\mathbb{R}^d$ and $z \in D$ a point with $\text{dist}(z, \partial D) > \epsilon$. Define the (**spherical**) **mean value** of f on the sphere $B_\epsilon(z)$ of radius ϵ about z:

$$\text{MV}(f, z, \epsilon) = \int_{|\omega - z| = \epsilon} f(\omega)\, ds(\omega)$$

where s is the surface measure (constant times the surface area) normalized s.t. $\text{MV}(1, z, \epsilon) = 1$. A function $f : D \to \mathbb{R}$ is **harmonic** if it is continuous and satisfies the *mean value property*: $MV(f, z, \epsilon) = f(z)$ for every $z \in D$ with $\text{dist}(z, \partial D) > \epsilon$.

Let $\Delta f = \sum_{j=1}^{d} \partial^2 f / \partial x_j^2$ be the *Laplacian* of f, then a function $f : D \to \mathbb{R}$ is harmonic iff it is C^2 and $\Delta f(z) = 0$ for all $z \in D$. When dimension $d = 1$, then harmonic is equivalent to linear, but when $d > 1$ the class of harmonic functions is larger.

2.8 Dirichlet problem

The Dirichlet problem asks: let $D \subset \mathbb{R}^d$ be a bounded domain and $F : \partial D \to \mathbb{R}$ a continuous function, can we find an extension $f : \overline{D} \to \mathbb{R}$ of F such that:

(i) $f = F$ on ∂D;

(ii) f is continuous on $\overline{D}$;

(iii) $\Delta f(x) = 0$ for all $x \in D$.

We first show that if exists, such f is unique:

Let B_t be a d-dimensional BM and $\tau = \inf\{t \geq 0 : B_t \in \partial D\}$. Define $f(x) = \mathbf{E}^x[F(B_\tau)]$, then $f(x)$ is harmonic and continuous on D, and satisfies $f = F$ on ∂D. The only thing missing from f being a solution to the Dirichlet problem is the continuity on ∂D. It turns out, f is only continuous on all the regular points $z \in \partial D$.

Let X_t be a (time-homogeneous) Markov process taking values in $\mathbb{R}^d$, the **infinitesimal generator** L is the operator is defined by

$$Lg(x) = \lim_{t\downarrow 0} \frac{\mathbf{E}^x[g(X_t)] - g(x)}{t}.$$

Proposition 2.14. *Let B_t be a standard BM in $\mathbb{R}^d$ and f a C^2 function (that does not grow too fast at ∞), then $Lf(x) = \Delta f(x)/2$.*

Proof. WLOG assume that $x = 0$ and $f(0) = 0$. Since f is C^2, the Taylor expansion gives

$$f(B_t) = \sum_{j=1}^{d} \partial_j f(0) B_t^j + \frac{1}{2} \sum_{j=1}^{d} \sum_{k=1}^{d} \partial_{jk} f(0) B_t^j B_t^k + o(|B_t|^2).$$

Hence $Lf(x) = \lim_{t \downarrow 0} (\mathbf{E}^x[f(B_t)] - f(x))/t = ?$. □

Remark. If the BM B_t has mean $\mu \in \mathbb{R}^d$ and covariance matrix Γ, then

$$Lf(x) = \sum_{j=1}^{d} \mu_j \partial_j f(x) + \frac{1}{2} \sum_{j=1}^{d} \sum_{k=1}^{d} \partial_{jk} f(x) \Gamma_{jk}$$

Proposition 2.15. *Let B_t be a standard d-dimensional BM and $f : \mathbb{R}^d \to \mathbb{R}$ be bounded. Then for $t > 0$, $\varphi(t, x) = \mathbf{E}^x[f(B_t)]$ satisfies the heat equation*

$$\partial_t \varphi(t, x) = \frac{1}{2} \Delta_x \varphi(t, x).$$

Proof. A Brownian motion B_t starting at x has density $p_t(x, y) = (2\pi t)^{-d/2} e^{-|y-x|^2/2t}$, where $\partial_t p_t = \Delta_x p_t(x, y)/2$. Since f is bounded, we can exchange integral and derivative:

$$\begin{aligned}
\partial_t \varphi(t, x) &= \partial_t \int_{\mathbb{R}^d} f(y) p_t(x, y)\, dy \\
&= \int_{\mathbb{R}^d} f(y) \partial_t p_t(x, y)\, dy \\
&= \frac{1}{2} \int_{\mathbb{R}^d} f(y) \Delta_x p_t(x, y)\, dy \\
&= \frac{1}{2} \Delta_x \int_{\mathbb{R}^d} f(y) p_t(x, y)\, dy = \frac{1}{2} \Delta_x \varphi(t, x).
\end{aligned}$$

□

Let B_t be a standard d-dimensional BM. For $d \geq 3$, the **Green's function** $G(x, y)$ is defined by

$$\begin{aligned}
G(x, y) &= \lim_{\epsilon \downarrow 0} \mathbf{E}^x \left[\frac{1}{V_\epsilon} \int_0^\infty \mathbf{1}_{\{|B_t - y| \leq \epsilon\}}\, dt \right] \\
&= \lim_{\epsilon \downarrow 0} \frac{1}{V_\epsilon} \int_0^\infty \int_{|y-z| \leq \epsilon} p_t(x, z)\, dz dt \\
&= \int_0^\infty p_t(x, y)\, dt \\
&= \int_0^\infty \frac{1}{(2\pi t)^{d/2}} \exp\left\{ \frac{-|y-x|^2}{2t} \right\} dt = \frac{\Gamma(d/2 - 1)}{2\pi^{d/2}} |x - y|^{2-d},
\end{aligned}$$

where V_ϵ is the (d-dimensional) volume of the ϵ-ball. Intuitively, $G(x, y)$ represents the expected amount of times spent in y of a standard Brownian motion starting at x. $g_y(x) = G(x, y)$ is harmonic for $x \neq y$. Also define

$$G(x) = \int_0^\infty \frac{1}{(2\pi t)^{d/2}} \exp\left\{ \frac{-|x|^2}{2t} \right\} dt$$

and $G_D(x, y)$ to be the expected amount of time spent in y *before leaving* D of a standard BM starting at x. Then $G(x, y) = G_D(x, y) + \mathbf{E}^x[G(B_\tau, y)]$ where $\tau = \inf\{t : B_t \in \partial D\}$.

Proposition 2.16. *Let $g(y) = G_D(x,y)$, then*

(i) g is harmonic on $D \setminus \{x\}$;

(ii) $g(y) \to 0$ for $y \to z \in \partial D$ and z regular;

(iii) $G_D(x,y) = G_D(y,x)$.

For $d \leq 2$, the recurrence property of BM makes $G(x) = \infty$. Hence we redefine, for $d = 2$, the **potential kernel**

$$G(x) = \int_0^\infty [p_t(0,z) - p_t(o,x)]\,dt = \int_0^\infty \frac{1}{2\pi t}\left[e^{-1/2t} - e^{-|x|^2/2t}\right]dt = -\frac{1}{\pi}\log|x|$$

where we take $z = (0,1)$. This is a radially symmetric harmonic function on $\mathbb{R}^2 \setminus \{0\}$, and carries the intuitive meaning of "time spent in x compared to the time spent on the unit circle. Hence $G(x) = 0$ for $|x| = 1$. When t is large, $e^{-1/2t} - e^{-|x|^2/2t} = \frac{|x|^2-1}{2t} + O(1/t^2)$, hence $G(x) < \infty$ for $x \neq 0$. If we choose $z \neq (0,1)$, then $G(x)$ will differ from $-\log|x|/\pi$ by a constant.

Proposition 2.16 still works for this $G_D(x,y)$ defined as such in $d = 2$, where

$$G_D(x,y) = \mathbf{E}^x\left[\frac{\log|B_\tau - z|}{\pi}\right] - \frac{\log|x-z|}{\pi}.$$

Let $D \subset \mathbb{R}^2$ be a bounded domain. Let $G_D(x,y) = \int_0^\infty p_t^D(x,y)\,dt$ where $p_t^D(x,y)$ is the density of BM starting at x and killed when reaching the boundary, then for $f : D \to \mathbb{R}$ a continuous function,

$$\mathbf{E}^x\left[f(B_t)\mathbf{1}_{\{t<T\}}\right] = \int_D p_t^D(x,y)f(y)\,dy.$$

Also, $p_t^D(x,y)$ satisfies the heat equation: $\partial_t p_t^D(x,y) = \Delta_x p_t^D(x,y)/2$, and $p_t^D(x,y) = p_t^D(y,x)$. We say that $p_t^D(x,y)$ is the solution to the heat equation with initial condition $p_0(x,y) = \delta_{xy}$. Also

$$p_t^D(x,y) = p_t(x,y) - \mathbf{E}^x\left[p_{t-\tau}(B_\tau, y)\mathbf{1}_{\{t>T\}}\right].$$

3 Stochastic Calculus

3.1 Construction

Consider the differential equation

$$dX_t = R_t\,dt + A_t\,dB_t$$

where B_t is a BM, and X_t and R_t are random variables. This equation carries the meaning "at time t, X_t looks *locally* like a BM(R_t, A_t^2). We can not differentiate both sides by dt since BM is non-differentiable. But we can consider

$$X_t = X_0 + \int_0^t R_s\,ds + \int_0^t A_s\,dB_s.$$

It remains to make sense of the last term.

A $\{\mathcal{F}_t\}$-adapted-process A_t is **simple** if there exists $0 = t_0 < t_1 < \cdots < t_n = \infty$ and L^2-random variables $Y_0, \cdots, Y_n$ such that Y_j is $\mathcal{F}_{t_j}$-measurable and $A_t = Y_j$ for $t \in [t_j, t_{j+1})$.

Let B_t be a Let B_t be a standard BM and A_t be a $\mathcal{F}_t$-adapted standard BM. The (Itô) **stochastic integral** of A_t is defined by

$$Z_t = \int_0^t A_s\,dB_s = \sum_{i=o}^{j-1} Y_i(B_{t_{i+1}} - B_{t_i}) + Y_j(B_t - B_{t_j})$$

for $t \in [t_j, t_{j+1}]$.

Proposition 3.1. *The stochastic integral defined satisfies the following properties:*

(i) Z_t is adapted to $\{\mathcal{F}_t\}$;

(ii) (linearity) let a and b be constants, and C_t be a simple process, then

$$\int_0^t (aA_s + bC_s)\,dB_s = a\int_0^t A_s\,dB_s + b\int_0^t C_s\,dB_s;$$

also if $r \in [0, t]$, then

$$\int_0^t A_s\,dB_s = \int_0^r A_s\,dB_s + \int_r^t A_s\,dB_s;$$

(iii) (martingale) Z_t is an L^2-martingale;

(iv) (variance isometry)

$$\mathrm{Var}(Z_t) = \mathbf{E}(Z_t^2) = \int_0^t \mathbf{E}(A_s^2)\,dB_s;$$

(v) (continuity) w.p.1, $t \mapsto Z_t$ is a continuous function.

Proof. f □

Proposition 3.2. *Let A_t be a bounded, continuous, and $\{\mathcal{F}_t\}$-adapted process. Then for any t_0, there exists a sequence of bounded simple processes $A_t^{(n)}$ converging to A_t in the sense that*

$$\lim_{n\to\infty} \mathbf{E}\left[\int_0^{t_0} (A_t - A_t^{(n)})^2\,dt\right] = 0.$$

Hence we extend the definition of stochastic integral for bounded continuous (adapted) process A_t:

$$Z_t = \int_0^t A_s \, dB_s = \lim_{n\to\infty} \int_0^t A_s^{(n)} \, dB_s.$$

Remark. properties, continuity

A $\{\mathcal{F}_t\}$-adapted process $A_s(\omega)$ is **progressively measurable** if $A_s(\omega)$ is measurable in $\mathcal{B}([0,t])\otimes\mathcal{F}_t$ measurable for $s \in [0,t]$.

3.2 Itô's formula

Proposition 3.3 (Itô's formula). *Let B_t be a standard BM and $f : \mathbb{R} \to \mathbb{R}$ be C^2, then for every t,*

$$f(B_t) - f(B_0) = \int_0^t f'(B_s) \, dB_s + \frac{1}{2} \int_0^t f''(B_s) \, ds. \tag{$*$}$$

Proof. Since both sides of $(*)$ are continuous, it suffices to show $(*)$ for dyadics $t \in \mathfrak{D}$. We do the case for $t = 1$, the other $t \in \mathfrak{D}$ will be similar. Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition. Then

$$f(B_t) - f(B_0) = \sum_{j=1}^n \left(f(B_{t_j}) - f(B_{t_{j-1}}) \right).$$

Assume that there exists $C < \infty$ such that $|f''| < C$. By Taylor expansion, we have

$$\frac{1}{2} m_{j,n} (B_{t_j} - B_{t_{j-1}})^2 \le \left| f(B_{t_j}) - f(B_{t_{j-1}}) - f'(B_{t_{j-1}})(B_{t_j} - B_{t_{j-1}}) \right| \le \frac{1}{2} M_{j,n} (B_{t_j} - B_{t_{j-1}})^2$$

where $M_{j,n} = \sup_{z \in [B_{t_{j-1}}, B_{t_j}]} f''(z)$ and $m_{j,n} = \inf_{z \in [B_{t_{j-1}}, B_{t_j}]} f''(z)$. Refine the partition so that

$$\sum_{j=1}^n f'(B_{t_{j-1}})(B_{t_j} - B_{t_{j_1}}) \to \int_0^1 f'(B_s) \, dB_s, \text{ and}$$

$$\sum_{j=1}^n \frac{1}{2} m_{j,n} (B_{t_j} - B_{t_{j-1}})^2, \sum_{j=1}^n \frac{1}{2} M_{j,n} (B_{t_j} - B_{t_{j-1}})^2 \to \frac{1}{2} \int_0^1 f''(B_s) \, ds.$$

If $|f|$ is not bounded, then let $T_K = \inf\{t : |f''(B_t)| \ge K\}$. Then Itô's formula holds for $t \wedge T_K$. Now let $K \to \infty$. □

Remark. Itô's formula is often written in the differential form:

$$df(B_t) = f'(B_t) \, dB_t + \frac{1}{2} f''(B_t) \, dt.$$

Intuitively, the process $Y_t = f(B_t)$ evolves like a Brownian motion with drift $f''(B_t)/2$ and variance $f'(B_t)^2$ locally at time t.

Proposition 3.4 (Itô's formula, II). *Let $f(t,x)$ be C^1 in t and C^2 in x, then*

$$f(t, B_t) - f(0, B_0) = \int_0^t \left(\partial_t f(s, B_s) + \frac{1}{2} \partial_x^2 f(s, B_s) \right) ds + \int_0^t \partial_x f(s, B_s) \, dB_s$$

Proof. Again, it suffices to show for $t = 1$. Observe that

$$f(t_j, B_{t_j}) - f(t_{j-1}, B_{t_{j-1}}) = f(t_j, B_{t_j}) - f(t_{j-1}, B_{t_j}) + f(t_{j-1}, B_{t_j}) - f(t_{j-1}, B_{t_{j-1}}).$$

Hence taking sum and refining the partition

$$\sum_{j=1}^{n} f(t_j, B_{t_j}) - f(t_{j-1}, B_{t_{j-1}}) = \sum_{j=1}^{n} \partial_t f(t_j, B_{t_j})(t_j - t_{j-1}) + ...$$

□

Let $f(t,x) = e^{at+bx}$ and $X_t = f(t, B_t)$, then

$$dX_t = X_t \left[\left(a + \frac{b^2}{2} \right) dt + b \, dB_t \right].$$

Hence solution to the SDE $dX_t = X_t(\mu \, dt + b \, dB_t$ is the **geometric Brownian motion**

$$X_t = X_0 \exp\{(\mu - \sigma^2/2)t + \sigma B_t\}.$$

Proposition 3.5. *Let $\{X_t\}_{0 \le t \le 1}$ be a process satisfying $dX_t = R_t \, dt + A_t \, dB_t$, then the quadratic variation*

$$\langle X \rangle_t = \lim_{n \to \infty} \sum_{j=1}^{n} \left(X_{jt/n} - X_{(j-1)t/n} \right)^2$$

of X_t is given by

$$\left\langle \int_0^t A_s \, dB_s \right\rangle = \lim_{n \to \infty} \sum_{j=1}^{n} \left(\int_0^{tj/n} A_s \, dB_s \right)^2 = \lim_{n \to \infty} \sum_{j=1}^{n} \int_0^{tj/n} A_s^2 \, dB_s.$$

We write $d\langle X \rangle_t = A_t^2 \, dt$.

Proof. Let $j = 1$, then

$$\begin{aligned}
(X_{t/n} - X_0)^2 &= \left(\int_0^{t/n} R_s \, ds + \int_0^{t/n} A_s \, dB_s \right)^2 \\
&= \left(\int_0^{t/n} R_s \, ds \right)^2 + 2 \left(\int_0^{t/n} R_s \, ds \right) \left(\int_0^{t/n} A_s \, dB_s \right) + \left(\int_0^{t/n} A_s \, dB_s \right)^2 \\
&= O(1/n^2) + O(1/n)O(1/\sqrt{n}) + O(1/n).
\end{aligned}$$

There are n terms of this form in the sum. Hence as $n \to \infty$ the only term contributing to the sum is the last term of order $O(1/n)$. □

Proposition 3.6 (Itô's formula (III))**.** *Let X_t be the process that satisfies $dX_t = R_t \, dt + A_t \, dB_t$. Let $f(t,x)$ be C^1 in t and C^2 in x. Then*

$$\begin{aligned}
df(t.X_t) &= \partial_t f(t, X_t) \, dt + \partial_x f(t, X_t) \, dX_t + \frac{1}{2} \partial_{xx}^2 f(t, X_t) \, d\langle X \rangle_t \\
&= \left(\partial_t f(t, X_t) + R_t \partial_x f(t, X_t) + \frac{1}{2} A_t^2 \partial_{xx}^2 f(t, X_t) \right) dt + A_t \partial_x f(t, X_t) \, dB_t.
\end{aligned}$$

Diffusion

A **diffusion process** X_t is a process satisfying

$$dX_t = m(t, X_t)\, dt + \sigma(t, X_t)\, dB_t \tag{$*$}$$

with certain initial condition $X_0 = x_0$. We can simulate a diffusion process using stochastic Euler's method: take $x_0 = 0$ and small Δt ad define recursively

$$X_{(k+1)\Delta t} = X_{k\Delta t} + m(k\Delta t, X_{k\Delta t})\Delta t + \sigma(k\Delta t, X_{k\Delta t})\sqrt{\Delta t} N,$$

where $N \sim N(0,1)$. Suppose f is nice enough (exchanging limit and integral) so that the generator

$$Lf(x) = \lim_{t\downarrow 0} \frac{\mathbf{E}^x[f(X_t)] - f(x)}{t}$$

is defined. For simplicity assume time homogeneity ($m(t,x) = m(x)$ and $\sigma(t,x) = \sigma(x)$) with σ and m both continuous and bounded. By Itô's formula

$$\mathbf{E}^x[f(X_t)] = \mathbf{E}\left[\int_0^t \left(m(X_s)f'(X_s) + \frac{1}{2}\sigma^2(X_s)f''(X_s)\right) ds + \int_0^t \sigma(X_s)f'(X_s)\, dB_t\right].$$

The last term is a martingale with expectation 0. Hence the generator is given by

$$Lf(x) = m(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x).$$

Now we want to solve the SDE given by $(*)$ with initial condition $X_0 = x_0$ to show that diffusion process X_t actually exists, under the assumption that m and σ are uniformly Lipschitz in x:

$$|m(t,x) - m(t,y)| \le \beta|x-y| \text{ and } |\sigma(t,x) - \sigma(t,y)| \le \beta|x-y|$$

for some $\beta > 0$. We use the stochastic version of Picard iteration method from ODE. Let $X_t^{(0)} = x_0$ and define recursively

$$X_t^{(n+1)} = x_0 + \int_0^t m(s, X_s^{(n)})\, ds + \int_0^t \sigma(s, X_s^{(m)})\, dB_s.$$

For simplicity assume $0 \le t \le 1$. We want to show that the L^2-norm is bounded:

$$\mathbf{E}\left[\left|X_t^{(n+1)} - X_t^{(n)}\right|^2\right] \le C\frac{\beta^{2n}t^{n+1}}{(n+1)!}, \tag{$**$}$$

so that the L^2-limit $X_t = \lim_{n\to\infty} X_t^{(n)}$ exists for each t. We can choose $t \in \mathfrak{D} \cap [0,1]$ first and then extend it. Fix t, we show $(**)$ by induction:

$$\begin{aligned}
\text{LHS} &\le 2\mathbf{E}\left[\left(\int_0^t \left(m(X_s^{(n)}) - m(X_s^{(n-1)})\right) ds\right)^2\right] + 2\mathbf{E}\left[\left(\int_0^t \left(\sigma(X_s^{(n)}) - \sigma(X_s^{(n-1)})\right) dB_s\right)^2\right] \\
&\le 2\mathbf{E}\left[\int_0^t \left(m(X_s^{(n)}) - m(X_s^{(n-1)})\right)^2 ds\right] + 2\mathbf{E}\left[\int_0^t \left(\sigma(X_s^{(n)}) - \sigma(X_s^{(n-1)})\right)^2 dB_s\right] \\
&\le 4\mathbf{E}\left[\int_0^t \beta^2\left|X_s^{(n)} - X_s^{(n-1)}\right|^2 ds\right] \\
&\le 4\beta^2 \int_0^t C\frac{\beta^{2(n-1)}s^n}{n!}\, ds = C\frac{\beta^{2n}t^{n+1}}{(n+1)!}.
\end{aligned}$$

A particular example of diffusion process is **Bessel process** X_t which satisfies

$$dX_t = \frac{a}{X_t}\, dt + dB_t$$

for $0 \le t < T = \min\{t : X_t = 0\}$, where $a \in \mathbb{R}$ and $X_0 = x_0 > 0$.

Proposition 3.7 (product rule)**.** *Let $dX_t = R_t\, dt + A_t\, dB_t$ and $dY_t = S_t\, dt + C_t\, dB_t$, then $d(X_tY_t) = X_t\, dY_t + Y_t\, dX_t + d\langle X, Y\rangle_t$, where the covariation term*

$$\langle X, Y\rangle_t = \lim_{n\to\infty} \sum_{j \le nt} (X_{j/n} - X_{(j-1)/n})(Y_{j/n} - Y_{(j-1)/n}) = \int_0^t A_s C_s\, ds.$$

Proof. Sketch.

$$\begin{aligned} X_{t+\Delta t}Y_{t+\Delta t} - X_tY_t &= (X_{t+\Delta t} - X_t)Y_{t+\Delta t} + X_tY_{t+\Delta t} - X_tY_t \\ &= (X_{t+\Delta t} - X_t)Y_t + (X_{t+\Delta t} - X_t)(Y_{t+\Delta t} - Y_t) + X_t(Y_{t+\Delta t} - Y_t) \\ &\approx Y_t\, dX_t + d(X_t)d(Y_t) + X_t\, dY_t \end{aligned}$$

where $d(X_t)d(Y_t)$ is nonzero. As for the covariation, notice that $(dB_t)^2 \approx dt$. □

In higher dimensions

Let $B_t = (B_t^1, \cdots, B_t^d)$ be a d-dimensional BM and $X_t = (X_t^1, \cdots, X_t^n)$ a vector process satisfying

$$dX_t^j = R_t^j\, dt + \sum_{k=1}^d A_t^{j,k}\, dB_t^k.$$

Since we have $\langle B^j, B^k\rangle_t = 0$ for $j \neq k$ and $\langle B^j, B^k\rangle_t = t$ when $j = k$, we have

$$\langle X^j\rangle_t = \sum_{k=1}^d \int_0^t (A_s^{j,k})^2\, ds, \quad \text{and} \quad \langle X^j, X^k\rangle_t = \sum_{i=1}^t A_s^{j,i} A_s^{k,i}\, ds.$$

Proposition 3.8 (Itô' formula, final form)**.** *Suppose the vector process $X_t = (X_t^1, \cdots, X_t^n)$ satisfies*

$$dX_t^j = R_t^j\, dt + \sum_{k=1}^d A_t^{j,k}\, dB_t^k.$$

Let $f(t, x) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ be C^1 in t and C^2 in x. Then

$$f(t, X_t) = f(0, X_0) + \int_0^t \partial_t f(s, X_s)\, ds + \sum_{j=1}^n \int_0^t \partial_j f(s, X_s)\, dX_s^j + \frac{1}{2} \sum_{j=1}^k \sum_{k=1}^n \partial_{jk}^2 f(s, X_s)\, d\langle X^j, X^k\rangle_s.$$

In particular if $n = d$ and $X_t = B_t$, then

$$df(t, B_t) = \nabla_x f(t, B_t) \cdot dB_t + \left[\partial_t f(t, B_t) + \frac{1}{2}\Delta_x f(t, B_t)\right] dt$$

Some applications

Martingale betting strategy, arbitrage: we want the process to reach value 1 at time $t = 1$. If it reaches 1 at $t = 1/2$ we simply stops betting. If it does not reach 1 at $t = 1/2$, then we bet more dollars so that the variance is big, and there is a positive probability or reaching 1 at $t = 3/4$. Repeat the observation at $t = 3/4$. The moral is: in a continuous-time setting we can always "win" within a given time, because we can bet infinitely many times.

European option: Let T (deterministic) be the call time and K be the strike price. Suppose the stock (asset) price X_t is a diffusion process

$$dX_t = m(t, X_t)\, dt + \sigma(t, X_t)\, dB_t.$$

The value of the option at time T is $F(X_T) = (X_T - K) \vee 0$. Let $r(t, x)$ be the interest (bond) rate and

$$R_t = \exp\left\{ \int_0^t r(s, X_s)\, ds \right\}.$$

One dollar now ($t = 0$) will be promised to value R_t dollars in the future at time t. Then

$$\varphi(t, x) = \mathbf{E}\left[\exp\left\{ -\int_t^T r(s, X_s)\, ds \right\} F(X_T) \mid X_t = x \right] = \mathbf{E}\left[(R_t/R_T) F(X_T) \mid X_t = x \right]$$

is a martingale since

$$M_t = \mathbf{E}\left[R_T^{-1} F(X_T) \mid \mathcal{F}_t \right] = R_T^{-1} \varphi(t, x)$$

is a martingale.

Change of measure

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space. Let B_t be a standard Brownian motion. Suppose that M_t satisfies $dM_t = A_t M_t\, dB_t$ and $M_0 = 1$. Then $M_t = e^{Y_t}$ is a only local martingale, where

$$Y_t = \int_0^t A_s\, dB_s - \frac{1}{2} \int_0^t A_s^2\, ds$$

by Itô's formula. Let $T_n = \inf\{t : M_t = n \text{ or } \langle M \rangle_t = n\}$ and $T = \lim_{n\to\infty} T_n$. Then $M_{t\wedge T_n}$ is a nonnegative martingale. Now let M_t be a nonnegative martingale. Define probability measure $\mathbb{P}^*$ by $\mathbb{P}^*_t(V) = \mathbf{E}[M_t \mathbf{1}_V]$ for $V \in \mathcal{F}_t$. That is, M_t is the Radon-Nikodym derivative $d\mathbb{P}^*/d\mathbb{P}$.

Lemma 3.9. *If one of the following holds, then M_s is a martingale for $0 \le s \le t$:*

(i) $\mathbf{E}(M_t) = 1$*;*

(ii) $\mathbb{P}^*\{T > t\} = 1$*;*

(iii) (Novikov's condition) $\mathbf{E}[\exp\{\langle Y \rangle_t / 2\}] < \infty$*.*

Theorem 3.10 (Girsanov's theorem)**.** *If M_t is a nonnegative martingale, then $dB_t = A_t\, dt + dW_t$ for $t < T$, where W_t is a standard Brownian motion with respect to $\mathbb{P}^*$.*

Proof. Firstly, $W_0 = 0$ and $t \mapsto W_t$ is continuous. Since $\mathbb{P}^* \ll \mathbb{P}$ on $\mathcal{F}_t$, we have $\langle W \rangle_t = t$. Hence by, it suffices to show that W_t is a $\mathbb{P}^*$-martingale. □

Remark. Heuristics of Girsanov's theorem

Example (Brownian motion tilted by B_t^1)**.** Let $M_t = B_t$ and $B_0 = 1$. Then M_t is a martingale and $dM_t = B_t^{-1} M_t \, dB_t$. Apply Girsanov's theorem we see that for $t < T$, $dB_t = B_t^{-1} \, dt + dW_t$. Hence B_t is a Bessel process with $a = 1$. Hence for fixed t, with $\mathbb{P}^*$-probability 1, $0 < B_s < \infty$ for $0 \le s \le t$ and $T = \infty$. But with respect to $\mathbb{P}$, B_s may reach 0. We have tilted the Brownian motion so that the weight of paths reaching 0 is 0.

Example (Brownian motion tilted by B_t^r)**.** Let $X_t = B_t^r$ and $B_t = 1$. Then M_t is a martingale and satisfies

$$dX_t = rB_t^{r-1} \, dB_t + \frac{r(r-1)}{2} B_t^{r-2} \, dt = \left(\frac{r}{B_t} \, dB_t + \frac{r(r-1)}{2B_t^2} \, dt \right) B_t^r$$

by Itô's formula. We use the following trick to get rid of the drift term: let $M_t = C_t X_t$ where

$$C_t = \exp\left\{ -\int_o^t \frac{r(r-1)}{2B_s^2} \, ds \right\}.$$

Then use product rule and the fact that $\langle C, X \rangle_t = 0$ to conclude that $dM_t = rB_t^{-1} M_t \, dB_t$. Apply Girsanov's theorem, we see that B_t is a Bessel process with $a = r$. When $r \ge 1/2$, M_t is a martingale since $\mathbb{P}^*\{T = \infty\} = 1$.

Let $dX_t = X_t(m \, dt + \sigma \, dB_t)$ where $m = m(t, X_t)$ and $\sigma = \sigma(t, X_t)$. We want to find a measure $\mathbb{P}^*$ under which the drift term disappears and X_t is a martingale. It suffices to show that $dB_t = -(m/\sigma) \, dt + dW_t$, where W_t is a standard Brownian motion under $\mathbb{P}^*$. Let $dM_t = (-m/\sigma) M_t \, dB_t$. To use Girsanov's theorem, we need that $M_t = e^{Y_t}$ is a nonnegative martingale.

Claim: M_t is a martingale if $\mathbf{E}(M_t) = 1$.

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Application to finance

Classification of Lévy process

A process $\{X_t\}_{t \ge 0}$ is a **Lévy process** if

(i) it has independent, identically distributed increments, and

(ii) $X_t \to X_0$ is probability as $t \downarrow 0$.

For example, Brownian motion (m, σ^2) and Poisson process (λ) are Lévy processes.

An example of Lévy process with discontinuous paths is $T_a = \inf\{t : B_t = a\}$ for B_t a standard Brownian motion. It has density $at^{-3/2}(2\pi)^{-1/2} \exp\{-a^2/2t\}$. It does not have continuous path, for let $a = \max\{B_t : t \in [0,1]\}$, then $\lim_{\epsilon \downarrow 0} T_{a+\epsilon} \ne T_a$. Similarly, let $B_t = (B_t^1, B_t^2)$ be a standard 2-dimensional Brownian motion. Then $Y_s = B_{T_s}^1$ is a Lévy process with Cauchy distribution $s^{-1}\pi(x^2 + s^2)$, where $T_s = \min\{t : B_t^2 = s\}$. It does not have continuous paths. It turns out that Brownian motion is the only Lévy process with continuous paths.

Our goal is to classify all Lévy process.

A random variable X has **infinitely divisible distribution** if for any $n \in \mathbb{Z}^+$ we can find a sequence of independent, identically distributed $Y_1, \cdots, Y_n$ such that $X = Y_1 + \cdots + Y_n$ in law.

Lemma 3.11. *If X_t is a Lévy process, then X_t has an infinitely divisible distribution for each t. Conversely, for any infinitely divisible distribution F, there is a Lévy process X_t such that X_1 has distribution F.*

Proof. Write $X_t = \sum_{j=1}^n (X_{jt/n} - X_{(j-1)t/n})$ as a telescoping sum. For the converse, □

Exercise from Wednesday Class (Due this week)

Let $D \subset \mathbb{R}^2$ be a bounded, simply connected domain containing the origin 0 and $\hat{D} = D \setminus \{0\}$. Let B_t be a standard Brownian motion starting at $B_0 = x \in \hat{D}$. Let $T_s = \inf\{t : |B_t| \le e^{-s}\}$, $T = \inf\{t : B_t = 0\}$, and $\tau = \inf\{t : B_t \notin D\}$.

(i) Let $G(x) = \lim_{s\to\infty} s\mathbb{P}^x\{T_s < \tau\}$. Show that $0 < G(x) < \infty$ for $x \in \hat{D}$.

Suppose D is contained in a circle C of radius R around the origin, then

$$\mathbb{P}^x\{T_s < \tau\} \le \mathbb{P}^x\{T_s < \tau_R\} = \frac{\log R - \log|x|}{\log R - \log e^{-s}},$$

where $\tau_R = \inf\{t : B_t \notin C\}$. Hence $\limsup_{s\to\infty} s\mathbb{P}^x\{T_s < \tau\} \le \log R + \log(1/|x|) < \infty$.

On the other hand, suppose D contains a circle c of radius r around the origin, then

$$\mathbb{P}^x\{T_s < \tau\} = \mathbb{P}^x\{\tau' < \tau\}\mathbb{P}^{\frac{r}{2}}\{T_s < \tau\} \ge \mathbb{P}^x\{\tau' < \tau\}\mathbb{P}^{\frac{r}{2}}\{T_s < \tau_r\} = K\frac{\log r - \log(r/2)}{\log r - \log e^{-s}},$$

where $K = \mathbb{P}^x\{\tau' < \tau\}$, $\tau' = \{t : |B_t| < r/2\}$, and $\tau_r = \inf\{t : B_t \notin c\}$. Hence

$$\liminf_{s\to\infty} s\mathbb{P}^x\{T_s < \tau\} \ge \liminf_{s\to\infty} sK\frac{\log r - \log(r/2)}{\log r - \log e^{-s}} = K(\log r - \log(r/2)) > 0.$$

(Monotonicity)

(ii) Show that G is harmonic in $\hat{D}$.

Choose any $x \in \hat{D}$. Let $\epsilon < e^{-s}$ and $\epsilon < |x|$. Then

$$s\mathbb{P}^x\{T_s < \tau\} = s\int_{|z-x|=\epsilon} \mathbb{P}^z\{T_s < \tau\}ds$$

satisfies the mean-value property, where s is the normalized surface measure. By monotonicity from part (i), we can exchange limit and integral. Hence $G(x) = \mathrm{MV}(G, x, \epsilon)$ satisfies the mean-value property, and is harmonic in $\hat{D}$.

(iii) Show that if $x_n \to z \in \partial D$, then $G(x_n) \to 0$.

(A conformal invariance argument)

(iv) Let $M_t = G(B_t)$ for $0 \le t < t \wedge T$. Show that M_t is a local martingale for $t < T \wedge \tau$.

Use Itô's formula: $dM_t = dG(B_t) = \nabla G(B_t) \cdot dB_t + \Delta G(B_t)\, dt/2 = \Delta G(B)_t \cdot dB_t$, since G is harmonic by part (ii). Hence M_t is a local martingale.

(v) Use Girsanov's theorem to find a new probability measure under which $T < \infty$ and $T < \tau$.

(vi) Is M_t a martingale?

Let $\{M_t\}_{t\geq 0}$ be a nonnegative discrete-time martingale with $M_0 = 1$. Let $M_\infty = \lim_{n\to\infty} M_n$. Let $\mathcal{F}_n$ be the σ-algebra generated by $\{M_0, \cdots, M_n\}$. Define measure $\mathbb{Q}_n$ on $\mathcal{F}_n$ by $\mathbb{Q}_n(V) = \mathbf{E}[M_n \mathbf{1}_V]$ for $V \in \mathcal{F}_n$. Let $\mathbb{Q}$ be the measure that is equal to $\mathbb{Q}_n$ when restricted to $\mathcal{F}_n$. Show that if $\mathbb{Q}\{M_\infty < \infty\} = 1$ or $\mathbb{Q}\{\sup_n M_n < \infty\} = 1$, then $\mathbf{E}[M_\infty] = 1$.