Martingale Central Limit Theorem. Let $\{l_t^N = \sum_{n=1}^t X_n^N : 1 \le t \le t_N\}_{N \ge 1}$ be an array of zero-mean discrete martingales. Then $l_{t_N}^N \to N(0, \sigma^2)$ in law if

- **(i)** $[l^N]_{t_N} = \sum_{n=1}^{t_N} (X_n^N)^2 \to \sigma^2$ in probability, for some constant σ^2, and
- **(ii)** $\mathbf{E} \max_{1 \le n \le t_N} \left| X_n^N \right| \to 0$.

Proof. We first modify the martingale while preserving **(i)** and **(ii)**. We then define a product martingale and work on it until we can apply Levy's convergence theorem.

Step one. Without loss of generality suppose that $\sup_N \{\max_{1 \le n \le t_N} \left| X_n^N \right|\} \le C$ for some constant $C > 0$. Define stopping time

$$\tau_N = t_N \wedge \min\{1 \le t \le t_N : [l^N]_t > \sigma^2 + 1\},$$

then $\mathbf{P}(\tau_N \ne t_N) \to 0$ because of **(i)**, and hence $[l_{t \wedge \tau_N}]_{t_N} \to \sigma^2$.

Embed discrete martingale $(l_t^N)_{1 \le t \le t_N}$ into a continuous martingale $(\bar{l}_t^N)_{1 \le t \le t_N}$ with continuous sample paths [Heath (1977)] and define similarly stopping time

$$\sigma_N = t_N \wedge \min\{1 \le t \le t_N : |\bar{l}_t^N - \bar{l}_{\lfloor t \rfloor}^N| \ge C\}.$$

By Jensen's inequality and note that $\bar{l}_{\lfloor \sigma_N \rfloor}^N$ is predictable (measurable) at $\bar{l}_{\sigma_N}^N$,

$$\left| \bar{l}_{\sigma_N}^N - \bar{l}_{\lfloor \sigma_N \rfloor}^N \right| = \left| \mathbf{E} \left(\bar{l}_{\sigma_N}^N - \bar{l}_{\lfloor \sigma_N \rfloor}^N \mid \bar{l}_{\sigma_N}^N \right) \right| \le \mathbf{E} \left(\left| \bar{l}_{\sigma_N}^N - \bar{l}_{\lfloor \sigma_N \rfloor}^N \right| \mid \bar{l}_{\sigma_N}^N \right) \le \mathbf{E} \left(\left| \bar{l}_{\lceil \sigma_N \rceil}^N - \bar{l}_{\lfloor \sigma_N \rfloor}^N \right| \mid \bar{l}_{\sigma_N}^N \right).$$

Hence by **(ii)**

$$\begin{aligned} \mathbf{P}(\sigma_N \ne t_N) &= \mathbf{P} \left(\left| \bar{l}_{\sigma_N}^N - \bar{l}_{\lfloor \sigma_N \rfloor}^N \right| \ge C \right) \\ &\le \frac{1}{C} \mathbf{E} \left| \bar{l}_{\sigma_N}^N - \bar{l}_{\lfloor \sigma_N \rfloor}^N \right| \\ &\le \frac{1}{C} \mathbf{E} \left| \bar{l}_{\lceil \sigma_N \rceil}^N - \bar{l}_{\lfloor \sigma_N \rfloor}^N \right| \le \frac{1}{C} \max_{1 \le n \le t_N} \left| X_n^N \right| \to 0, \end{aligned}$$

which implies that $\mathbf{E} \max_{1 \le n \le t_N} \left| l_{t \wedge \sigma_N}^N - l_{(t \wedge \sigma_N)-1}^N \right| \to 0$. Now replace $(l_t^N)_{1 \le t \le t_N}$ by the modified martingale $(l_t^{N,C})_{1 \le t \le t_N} = (\bar{l}_{t \wedge \sigma_N \wedge \tau_N}^N)_{1 \le t \le t_N}$ which satisfies in addition to **(i)** and **(ii)**,

- **(iii)** $[l^{N,C}]_{t_N} \le \sigma^2 + 1 + C^2$ and
- **(iv)** $\left| X_n^{N,C} \right| \le C$

for any N and n, where $X_n^{N,C} = l_n^{N,C} - l_{n-1}^{N,C}$.

Step two. Construct product martingale $m_t^{N,C}(\theta) = \prod_{n=1}^t (1 + \theta X_n^{N,C})$, then for $|\theta| < C^{-1}$ expand complex logarithm to get $\log m_{t_N}^{N,C}(\theta) = \sum_{p \ge 1} \frac{1}{p} \theta^p [l^{N,C}]_{t_N}^p$ where $[l^{N,C}]_t^p = \sum_{n=1}^{t_N} (X_n^{N,C})^p$ is the p-variation. Since $\max_{1 \le n \le t_N} |X_n^{N,C}| \to 0$ as $N \to \infty$,

$$\begin{aligned} \left| \log m_{t_N}^{N,C}(\theta) - \left(\theta l_{t_N}^{N,C} - \frac{1}{2} \theta^2 [l^{N,C}]_{t_N} \right) \right| &\le \sum_{p \ge 3} \frac{1}{p} \theta^p [l^{N,C}]_{t_N}^p \\ &\le [l^{N,C}]_{t_N} \sum_{p \ge 3} \frac{1}{p} \theta^p \left(\max_{1 \le n \le t_n} \left| X_n^{N,C} \right| \right)^{p-2} \to 0 \end{aligned}$$

in probability. Observe that $l_{t_N}^{N,C}$ is bounded in L^2 as $\mathbf{E}(l_{t_N}^{N,C})^2 = \mathbf{E}[l^{N,C}]_{t_N} \leq \sigma^2 + 1 + C^2$, hence $l_{t_N}^{N,C}$ is uniformly integrable and thus tight. But $[t^{N,C}]_{t_N}$ is also tight, hence $\theta l_{t_N}^{N,C} - \theta^2[l^{N,C}]_{t_N}/2$ is tight, and

$$m_{t_N}^{N,C}(\theta) = (1 + o_p(1)) \exp\left\{\theta l_{t_N}^{N,C} - \frac{1}{2}\theta^2\sigma^2\right\}. \tag{$*$}$$

Let $\theta = ih$ where $h \in \mathbf{R}$ and $|\theta| = |h| < C^{-1}$. Similarly $m_{t_N}^{N,C}$ is bounded in L^2 as

$$\mathbf{E}\left|m_{t_N}^{N,C}\right|^2 = \mathbf{E}\prod_{n=1}^{t_N}\left(1 + h^2(X_{t_N}^{N,C})^2\right) \leq \mathbf{E}\exp\left\{h^2[l^{N,C}]_{t_N}\right\} \leq \mathbf{E}\exp\left\{h^2(\sigma^2 + 1 + C^2)\right\}$$

hence $m_{t_N}^{N,C}$ is uniformly integrable, and from $(*)$,

$$\mathbf{E}\exp\left\{\theta l_{t_N}^{N,C} - \frac{1}{2}\theta^2\sigma^2\right\} \to \mathbf{E}m_{t_N}^{N,C} = 1.$$

Hence $\mathbf{E}\exp\left\{ihl_{t_N}^{N,C}\right\} \to \exp\left\{-h^2\sigma^2/2\right\}$. But $l_{t_N}^{N,C} \neq l_{t_N}^{N}$ only when $t_N \neq \tau_N$ or $t_N \neq \sigma_N$, hence

$$\left|\mathbf{E}\exp\left\{ihl_{t_N}^{N,C}\right\} - \mathbf{E}\exp\left\{ihl_{t_N}^{N}\right\}\right| = \left|\mathbf{E}\exp\left\{ihl_{t_N}^{N,C}\right\} - \mathbf{E}\exp\left\{ihl_{t_N}^{N}\right\}\right|\mathbf{1}_{\{t_N \neq \tau_N \text{ or } t_N \neq \sigma_N\}} \to 0.$$

Hence $\mathbf{E}\exp\left\{ihl_{t_N}^{N}\right\} \to \exp\left\{-h^2\sigma^2/2\right\}$, and by Levy's convergence theorem $l_{t_N}^{N} \to N(0, \sigma^2)$ in law as desired.

□

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