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2 Homology

Homology is a covariant functor from the category of topological spaces with continuous maps to the category of abelian groups with group homomorphisms, with chain complexes as an intermediate step. The central property of homology groups, like the fundamental groups, is the homotopy invariance. But homology groups are easier to compute than homotopy groups, in general.

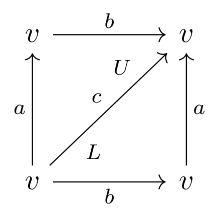
2.1 Basic concepts

Simplicial homology is built from Δ-complex structures and cellular homology is built upon CW complexes. These two homology theories are convenient in computation but require strict conditions on the space. Singular homology built from singular simplices is most used in proofs for its flexibility, but is less computable in general.

Simplicial homology A Δ-complex structure on space X is a collection of maps σ_n from the standard n-simplices Δ^n to X that decomposes X "nicely", with appropriate topology. The chain group is defined as $\Delta_n(X) = F^{ab}(\sigma_\alpha)$, and the boundary map is $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ by $\sigma_\alpha \mapsto \sum_i (-1)^i \sigma_\alpha|_{[v_0,\cdots,\hat{v}_i,\cdots,v_n]}$. Check that $\partial_n\partial_{n-1} = 0$, and thus $\Delta_.(X)$ forms a chain complex. The corresponding homology groups $H^\Delta_.(X)$ are the simplicial homology groups.

To calculate simplicial homology we need to subdivide for a Δ-complex structure.

$\mathbf{T}^2$:



$$\Longrightarrow \begin{cases} \text{2-simplices: } \sigma_U, \sigma_L \\ \text{1-simplices: } \sigma_a, \sigma_b, \sigma_c \\ \text{0-simplices: } \sigma_v \end{cases}$$

Thus we obtain the chain complex and the corresponding homology groups:

$$\begin{array}{ccccccccc} 0 & \xrightarrow{\partial_3} & \Delta_2(\mathbf{T}^2) & \xrightarrow{\partial_2} & \Delta_1(\mathbf{T}^2) & \xrightarrow{\partial_1} & \Delta_0(\mathbf{T}^2) & \xrightarrow{\partial_0} & 0 \\ & & \| & & \| & & \| & & \\ & & \mathbf{Z}^2 & & \mathbf{Z}^3 & & \mathbf{Z} & & \end{array} \Rightarrow \begin{cases} H^\Delta_2(\mathbf{T}^2) = \ker\partial_2 \cong \mathbf{Z} \\ H^\Delta_1(\mathbf{T}^2) \cong \mathbf{Z}^3/\mathbf{Z} \cong \mathbf{Z}^2 \\ H^\Delta_0(\mathbf{T}^2) = H^\Delta_3(\mathbf{T}^2) = 0 \end{cases}.$$

Similarly, we can do this for $\mathbf{RP}^2$ and Klein bottle K.

Singular homology A singular n-simplex in X is a map $\sigma : \Delta^n \to X$ that need not be "nice". The singular chain group $C_n(X)$ is the free abelian group generated by all singular n-simplices in X with the same boundary map as before. Similarly $C_.(X)$ forms a chain complex with corresponding singular homology groups $H_.(X)$.

For example, let X be nonempty and path-connected, then define augmented map $\varepsilon : C_0(X) \to \mathbf{Z}$ by $\sum_i n_i\sigma_i \mapsto \sum_i n_i$, which is surjective since X is nonempty. If we can show that $\ker\varepsilon = \operatorname{Im}\partial_1$, then $H_0(X) = \ker\partial_0/\operatorname{Im}\partial_1 = C_0(X)/\ker\varepsilon \cong \mathbf{Z}$. $\operatorname{Im}\partial_1 \subseteq \ker\varepsilon$ is easy, for the reverse inclusion construct appropriate singular 1-simplex.

Sometimes we use reduced homology $\widetilde{H}$ by attaching the augmented map ε, so that $\widetilde{H}_n(X) = H_n(X)$ for $n > 0$ and $\widetilde{H}_0(X) \oplus \mathbf{Z} = H_0(X)$. Now $\widetilde{H}_n(\text{pt}) = 0$ for all n.

For $A \subseteq X$, define $C_n(X, A) = C_n(X)/C_n(A)$ to be the n-th relative chain group. With the naturally inherited boundary map we obtain a complex, and thus relative homology groups $H_.(X, A)$. It's not hard to see that $H_n(X, A) = \widetilde{H}_n(X, A)$ if A is nonempty, and $H_n(X, A) = \widetilde{H}_n(X)$ for all n.

Now we've reached the heart of homology: homotopy invariance. For $f : X \to Y$ there is a chain map $f_\# : C_.(X) \to C_.(Y)$ by $\sigma \mapsto f\sigma$, and since $f_\#(\partial\sigma) = \partial f_\#(\sigma)$, the chain ladder commutes. Two chain maps $f_\#$ and $g_\#$ are homotopic if there exists $h_n : C_n(X) \to C_{n+1}(Y)$ such that $f_\# - g_\# = \partial'_{n+1}h_n + h_{n-1}\partial_n$ for all n. Chain map $f_\#$ induces $f_* : H_.(X) \to H_.(Y)$. As a functors, $(fg)_\# = f_\# g_\#$, $\mathrm{id}_\# = \mathrm{id}$, and $(fg)_* = f_* g_*$, $\mathrm{id}_* = \mathrm{id}$.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle h_{n-1}} & & \\ \cdots & \longrightarrow & C_n(Y) & \xrightarrow{\partial_n} & C_{n-1}(Y) & \longrightarrow & \cdots \end{array}$$

$$\begin{array}{ccccccc} & \cdots & \longrightarrow & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \\ & & \swarrow{\scriptstyle h_n} & {\scriptstyle f_\#}\downarrow\downarrow{\scriptstyle g_\#} & \swarrow{\scriptstyle h_{n-1}} & & \\ C_{n+1}(Y) & & \xrightarrow[\partial'_{n+1}]{} & C_n(Y) & \longrightarrow & \cdots & \end{array}$$

Theorem 2.1. *Let $f, g : X \to Y$. If $f \simeq g$, then $f_\# \simeq g_\#$, and thus $f_* = g_*$.*

Proof. We split the proof into two steps:

$f \simeq g \Rightarrow f_\# \simeq g_\#$. Let $i, i' : X \hookrightarrow X \times [0, 1]$ be the inclusions $x \mapsto (x, 0)$ and $x \mapsto (x, 1)$. Let $\Delta^n \times \{0\} = [u_0, \cdots, u_n]$ and $\Delta^n \times \{1\} = [v_0, \cdots, v_n]$, then we can subdivide $\Delta^n \times [0, 1]$ into $n + 1$ $(n + 1)$-simplices of the form $s_i = [u_0, \cdots, u_i, v_i, \cdots, v_n]$. Define $P_n = \sum_i (-1)^i s_i$, then by calculation $\partial P_n = \Delta^n \times \{1\} - \Delta^n \times \{0\} - P_n(\partial\Delta^n)$. Let $h_n : C_n(X) \to C_{n+1}(Y)$ by $\sigma \mapsto (\sigma \times \mathrm{id})_\#(P_n)$ where $\sigma : \Delta^n \to X$. Check that $i_\# \simeq i'_\#$ under h. Finally, notice that if $f \simeq g$ by F, then $f = Hi$ and $g = Hi'$.

$f_\# \simeq g_\# \Rightarrow f_* = g_*$. Let $z \in \ker \partial_n$, then $f_\#(z) - g_\#(z) = (\partial'_{n+1}h_n + h_{n-1}\partial_n)(z) = (\partial'_{n+1}h_n)(z)$ which is in $\mathrm{Im}\, \partial'_{n+1}$, hence $f_*([z]) = [f_\#(z)] = [g_\#(z)] = g_*([z])$. □

$C_n(X)$ is a much bigger group then $\Delta_n(X)$, but in terms of homology they are equivalent:

Theorem 2.2. *Let X be equipped with a Δ-complex structure, then the inclusion $\Delta_.(X) \hookrightarrow C_.(X)$ of chain complexes induces an isomorphism $H^\Delta_.(X) \cong H_.(X)$.*

Proof. If X is finite dimensional, then with excision and the fact that $H_n(\Delta^n, \partial\Delta^n) \cong \mathbf{Z}$ is generated by identity maps $\mathrm{id}_n : \Delta^n \to \Delta^n$, we find the same description of $H^\Delta_n(X^k, X^{k-1})$ and $H_n(X^k, X^{k-1})$: free abelian with basis the k-simplices of X when $n = k$ and trivial otherwise.

$$\begin{array}{ccccccccc} H^\Delta_{n+1}(X^k, X^{k-1}) & \longrightarrow & H^\Delta_n(X^{k-1}) & \longrightarrow & H^\Delta_n(X^k) & \longrightarrow & H^\Delta_n(X^k, X^{k-1}) & \longrightarrow & H^\Delta_{n-1}(X^{k-1}) \\ \| & & \text{induction}\| & & \downarrow & & \| & & \text{induction}\| \\ H_{n+1}(X^k, X^{k-1}) & \longrightarrow & H_n(X^{k-1}) & \longrightarrow & H_n(X^k) & \longrightarrow & H_n(X^k, X^{k-1}) & \longrightarrow & H_{n-1}(X^{k-1}) \end{array}$$

Apply the five lemma to the diagram above. If X is infinite-dimensional, we use the compactness argument □

Cellular homology CW structure is less strict then Δ-complex structure. Define cellular chain group to be $C^{CW}_n(X) = H_n(X^n, X^{n-1}) = F^{ab}(n\text{-cells of } X)$, and with boundary map $d_n = j_{n-1}\partial_n$

we obtain a cellular chain complex:

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & H_n(X^n) & \longrightarrow & H_n(X^{n+1}) \cong H_n(X) & \longrightarrow & 0 & & \\
 & & \partial_{n+1}\uparrow & \searrow^{j_n} & & & & & \\
\cdots & \longrightarrow & H_{n+1}(X^{n+1}, X^n) & \xrightarrow{d_{n+1}} & H_n(X^n, X^{n-1}) & \xrightarrow{d_n} & H_{n-1}(X^{n-1}, X^{n-2}) & \longrightarrow & \cdots . \\
 & & & & \downarrow\partial_n & \nearrow_{j_{n-1}} & & & \\
 & & & & H_{n-1}(X^{n-1}) & & & & \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

The corresponding homology groups $H_\cdot^{CW}(X)$ are the cellular homology groups.

Above we assumed the following observations when X is a CW-complex:

(i) $H_n(X^k, X^{k-1}) \cong \begin{cases} \bigoplus \mathbf{Z} \text{ (one for each } n\text{-cell of } X), & n = k \\ 0, & \text{else} \end{cases}.$

(ii) $i_* : H_k(X^k) \to H_k(X)$ induced by $i : X^k \hookrightarrow X$ is surjective.

(iii) $H_n(X^k) \cong \begin{cases} H_n(X) \text{ (induced by } i : X^k \hookrightarrow X), & n < k \\ 0, & n > k \end{cases}.$

(i) follows from $H_n(X^k, X^{k-1}) \cong \widetilde{H}_n(X^k/X^{k-1}) \cong \widetilde{H}_n(\vee_i \mathbf{S}^k)$ since (X^k, X^{k-1}) is a good pair. (ii) and (iii) when X is finite-dimensional follows from considering the exact sequence of (X^k, X^{k-1}) and induction on $H_k(X^0) = 0$ when $k > 0$. When X is infinite-dimensional recall that

Now we make explicit of the boundary maps d_n, with the help of mapping degree.

Observe that $d_{n+1} : H_{n+1}(X^{n+1}, X^n) \to H_n(X^n, X^{n-1})$ sends $e_\alpha^{n+1} \mapsto \sum d_{\alpha\beta} e_\beta^n$. The claim is that $d_{\alpha\beta} = \deg f_{\alpha\beta}$ where $f_{\alpha\beta} = q_\beta \circ \varphi_\alpha$, with $\varphi_\alpha : \partial\mathbf{D}^n \to X$ the attaching map and $q_\beta : X \to \mathbf{S}_\beta^n$ collapsing all of X except for e_β^n.

$$
\begin{array}{ccccc}
\widetilde{H}_{n+1}(\mathbf{D}_\alpha^{n+1}, \partial\mathbf{D}_\alpha^{n+1}) & \xrightarrow{\partial} & \widetilde{H}_n(\partial\mathbf{D}_\alpha^{n+1}) & \xrightarrow{f_{\alpha\beta*}} & \widetilde{H}_n(\mathbf{S}_\beta^n) \\
\downarrow \Phi_{\alpha*} & & \downarrow \varphi_{\alpha*} & \nearrow^{q_{\beta*}} & \uparrow q_{\beta_2*} \\
\widetilde{H}_{n+1}(X^{n+1}, X^n) & \xrightarrow{\partial} & \widetilde{H}_n(X^n) & \xrightarrow{q_{\beta_1*}} & \widetilde{H}_n(X^n/X^{n-1}) \\
 & \searrow_{d_{n+1}} & & & \| \\
 & & & & \widetilde{H}_n(X^n, X^{n-1})
\end{array}
$$

Take a generator $[\mathbf{D}^{n+1}] \in \widetilde{H}_{n+1}(\mathbf{D}_\alpha^{n+1}, \partial\mathbf{D}_\alpha^{n+1})$, ∂ sends it to a generator in $\widetilde{H}_n(\partial\mathbf{D}_\alpha^{n+1})$ which has the image $\deg f_{\alpha\beta}$ under $f_{\alpha\beta*}$. On the other hand, the characteristic map (extension of attaching map) $\Phi_{\alpha*}$ sends $[\mathbf{D}^{n+1}]$ to $[e_\alpha^{n+1}]$, and d_{n+1} further sends it to $\sum d_{\alpha\beta} e_\beta^n$, which projects to the β^{th} factor $d_{\alpha\beta}$ by q_{β_2*}. Since the diagram commutes, $d_{\alpha\beta} = \deg f_{\alpha\beta}$.

Theorem 2.3. *The inclusion $C_\cdot^{CW}(x) \hookrightarrow C_\cdot(X)$ induces an isomorphism $H_\cdot^{CW}(X) \cong H_\cdot(X)$.*

Proof. Since j_n is injective, $\operatorname{Im}\partial_{n+1} = \operatorname{Im} d_{n+1}$ and $H_n(X^n) = \operatorname{Im} j_n$. By exactness $\operatorname{Im} j_n = \ker \partial_n$, and since j_{n-1} is injective, $\ker\partial_n = \ker d_n$. Together we have $H_n^{CW}(X) \cong H_n(X^n)/\operatorname{Im}\partial_{n+1}$, which by exactness is precisely $H_n(X)$. □

Now we no longer have to subdivide for a Δ-complex structure like we did when computing the simplicial homology of $\mathbf{T}^2$.

$\mathbf{M}_g$, $\mathbf{RP}^n$, for $\mathbf{CP}^n$: recall that $\mathbf{CP}^n = \mathbf{S}^{2n+1}/\sim$ where $v \sim \lambda v$ when $|\lambda| = 1$. The "upper hemisphere" of $\mathbf{CP}^n$ consists of points $(\omega, (1-|\omega|^2)^{1/2})$ where $\omega \in \mathbf{C}^n \cong \mathbf{D}^{2n}$, the boundary of which corresponds to $(\omega, 0)$ with the identification of $\mathbf{CP}^{n-1}$. Hence inductively $\mathbf{CP}^n$ is obtained from $\mathbf{CP}^{n-1}$ by attaching a $2n$-cell: $\mathbf{CP}^n = e_0 \cup e_2 \cup \cdots \cup e_{2n}$. Hence the cellular chain complex is an alternation between 0 and $\mathbf{Z}$ with trivial boundary maps.

$$H_k(\mathbf{CP}^n) \cong \begin{cases} \mathbf{Z}, & k = 0, 2, \cdots, 2n \\ 0, & \text{else} \end{cases}.$$

Axioms for homology In general a

2.2 Tools

Zig-zag lemma The zig-zag lemma provides a way of constructing long exact sequence of homology groups from short exact sequence of chain complexes.

Theorem 2.4 (zig-zag lemma)**.** *A short exact sequence of chain complexes*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_n(A) & \xrightarrow{i} & C_n(B) & \xrightarrow{j} & C_n(C) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_A} & & \downarrow{\scriptstyle \partial_B} & & \downarrow{\scriptstyle \partial_C} & & \\
0 & \longrightarrow & C_{n-1}(A) & \xrightarrow{i} & C_{n-1}(B) & \xrightarrow{j} & C_{n-1}(C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

induces a long exact sequence of homology groups

$$\cdots \longrightarrow H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \longrightarrow \cdots.$$

Proof. We construct the connecting homomorphism $\partial : H_{n+1}(C.) \to H_n(A.)$ as follows. Choose $[c] \in H_{n+1}(C.)$, then $c \in C_{n+1}$ and $\partial_C c = 0$. Since j is surjective, there exists $b \in B_{n+1}$ such that $j(b) = c$ and $j(\partial_B b) = \partial_C j(b) = 0$. Now since $\partial_B b \in \ker j \cong \operatorname{Im} i$, there exists $a \in A_n$ such that $i(a) = \partial_B b$. Finally, $i(\partial_A a) = \partial_B i(a) = \partial_B \partial_B b = 0$, and since i is injective, $\partial_A a = 0$. Hence let $\partial : [c] \mapsto [a]$, and check that it is well defined, i.e., the choices of $[c]$ and b does not matter.

$$\begin{array}{ccccccccc}
 & & & & B_{n+1} & \xrightarrow{j} & C_{n+1} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
 & & A_n & \xrightarrow{i} & B_n & \xrightarrow{j} & C_n & & \\
 & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{i} & B_{n-1} & & & &
\end{array}.$$

Now what's left is to check for exactness by diagram chasing. □

The zig-zag lemma is most used in space pairs (X, A):

$$\cdots \longrightarrow H_n(X, A) \xrightarrow{j_*} H_n(X) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X, A) \longrightarrow \cdots,$$

where the connecting homomorphism corresponds with the boundary map: $\partial[\alpha] = [\partial\alpha]$.

For example, one can easily compute

$$\widetilde{H}_k(\mathbf{D}^n, \partial\mathbf{D}^n) \cong \widetilde{H}_k(\mathbf{S}^{n-1}) \cong \begin{cases} \mathbf{Z}, & k = n \\ 0, & \text{else} \end{cases},$$

and deduce therefore the Brouwer fixed-point theorem: Every map $f : \mathbf{D}^n \to \mathbf{D}^n$ has a fixed point.

$$\begin{array}{ccc} \partial\mathbf{D}^n & \xrightarrow{i} & \mathbf{D}^n \\ & \searrow^{\text{id}} & \downarrow r \\ & & \partial\mathbf{D}^n \end{array} \implies \begin{array}{ccc} \widetilde{H}_{n-1}(\mathbf{S}^{n-1}) \cong \mathbf{Z} & \xrightarrow{i_*} & \widetilde{H}_{n-1}(\mathbf{D}^n) \cong 0 \\ & \searrow^{\text{id}} & \downarrow r_* \\ & & \mathbf{Z} \end{array}$$

Five lemma Another elementary yet useful tool from homological algebra.

Theorem 2.5 (five lemma)**.** *In the diagram bellow if the rows are exact, then* γ *is an isomorphism:*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow\alpha & & \| \beta & & \downarrow\gamma & & \| \delta & & \downarrow\varepsilon \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}.$$

Proof. β, δ surjective, ε injective $\Rightarrow$ γ surjective. β, δ injective, α surjective $\Rightarrow$ γ injective. □

Excision Excision is another fundamental property of homology, being one of the three axioms. It is also helpful in calculation. Excision theorem is remarkably easy when X has a Δ-complex structure with $A, X \setminus Z, A \setminus Z$ as Δ-subcomplexes:

$$\Delta_n(X \setminus Z) \longrightarrow \Delta_n(X) \longrightarrow \Delta_n(X, A), \qquad \varphi : \Delta_n(X \setminus Z) \to \Delta_n(X, A)$$

φ is surjective since a basis of $\Delta_n(X, A)$ is given by subcomplexes of $X \setminus A \subseteq X \setminus Z$. Hence φ induces an isomorphism $\Delta_n(X, A) \cong \Delta_n(X \setminus Z)/\ker\varphi = \Delta_n(X \setminus Z)/\Delta_n(A \setminus Z) = \Delta_n(X \setminus Z, A \setminus Z)$.

Theorem 2.6 (excision theorem)**.** *If* $\overline{Z} \subseteq A^\circ$, *then* $H_n(X, A) \cong H_n(X \setminus Z, A \setminus Z)$ *for all* n.

Proof. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Define $C_n^{\mathcal{U}}(X) \subseteq C_n(X)$ to be the subcomplex generated by n-simplices of X such that $\sigma(\Delta^n) \subseteq U_\alpha$ for some α. The boundary map inherited from $\partial : C_n(X) \to C_{n-1}(X)$ makes $(C_.^{\mathcal{U}}(X), \partial)$ into a chain complex.

By so-called barycentric subdivision $S : C.(X) \to C.(X)$ one can divide simplices so that each small simplex lies inside some U_α, and by showing that S is chain homotopic to the identity map, make sense of $H_n(C_.^{\mathcal{U}}(X)) \cong H_n(C.(X))$ for all n.

Now let $Y = X \setminus Z$ and $\mathcal{U} = \{Y, A\}$. We have $C_n(X \setminus Z)/C_n(A \setminus Z) = C_n(Y)/C_n(Y \cap A)$, which is generated by the simplices that lie in $X \setminus A$. Hence $C_n(Y)/C_n(Y \cap A) \cong C_n^{\mathcal{U}}(X)/C_n(A)$. Now we have the exact sequences of homology by the zig-zag lemma:

$$\begin{array}{ccccccccc} H_n(C.(A)) & \longrightarrow & H_n(C_.^{\mathcal{U}}(X)) & \longrightarrow & H_n(C_.^{\mathcal{U}}(X)/C.(A)) & \longrightarrow & H_{n-1}(C.(A)) & \longrightarrow & H_{n-1}(C_.^{\mathcal{U}}(X)) \\ \| & & \| & & \downarrow & & \| & & \| \\ H_n(C.(A)) & \longrightarrow & H_n(C.(X)) & \longrightarrow & H_n(C.(X)/C.(A)) & \longrightarrow & H_{n-1}(C.(A)) & \longrightarrow & H_{n-1}(C.(X)) \end{array}.$$

By the five lemma $H_n(C_.^{\mathcal{U}}(X)/C.(A)) \cong H_n(C.(X)/C.(A))$. Hence $H_n(X \setminus Z, A \setminus Z) \cong H_n(X, A)$. □

Equivalently, excision theorem says that if $X = A^\circ \cup B^\circ$, then $(B, A\cap B) \hookrightarrow (X, A)$ induces $H_n(B, A\cap B) \cong H_n(X, A)$ for all n. To see this let $Z = X \setminus B$ and $B = X \setminus Z$ for the converse.

Excision theorem tells us that for a "good" pair (X, A), i.e., $A \subseteq X$ is closed, nonempty, and deformation retract to some some neighborhood $V \subseteq X$ of A (e.g., CW pairs are "good"), then $H_n(X, A) \cong \widetilde{H}_n(X/A)$. Otherwise $\widetilde{H}_n(X, A) \cong H_n(X \cup \mathrm{Cone}_A, \mathrm{Cone}_A) \cong \widetilde{H}_n(X \cup \mathrm{Cone}_A)$ is always true for arbitrary pairs, where the first isomorphism is obtained by excising the cone tip.

$$\begin{array}{ccc}
H_n(X,A) & \xlongequal{\text{excision}} & H_n(X\setminus A, V\setminus A) \cong H_n(X,V) \\
\downarrow & & \| \\
H_n(X/A, A/A) & = & H_n(X/A \setminus A/A, V/A \setminus A/A) \cong H_n(X/A, V/A)
\end{array}$$

For example, $H_n(\Delta^n, \partial\Delta^n) \cong \mathbf{Z}$ is generated by identity maps $\mathrm{id}_n : \Delta^n \to \Delta^n$, a key ingredient in the proof of Theorem 2.2. When $n = 0$ the statement is trivial. We proceed by induction on n. Let Λ^n be $\partial\Delta^n$ setminus the last face (in the case of a triangle Δ^2, Λ^2 looks exactly like a triangle without the bottom side) and consider the exact sequence of the triple $(\Delta^{n+1}, \partial\Delta^{n+1}, \Lambda^{n+1})$

$$\begin{array}{ccccccc}
H_{n+1}(\Delta^{n+1}, \Lambda^{n+1}) & \longrightarrow & H_{n+1}(\Delta^{n+1}, \partial\Delta^{n+1}) & \xrightarrow{\partial} & H_n(\partial\Delta^{n+1}, \Lambda^{n+1}) & \longrightarrow & H_n(\Delta^{n+1}, \Lambda^{n+1}) \\
\| & & & & \text{"good"}\| & & \| \\
0 & & & & H_n(\Delta^n, \partial\Delta^n) & & 0
\end{array}$$

to conclude that ∂ is an isomorphism. Let $\mathrm{id}_{n+1} \in H_{n+1}(\Delta^{n+1}, \partial\Delta^{n+1})$, then

$$\partial \mathrm{id}_{n+1} = [\sum_k (-1)^k \mathrm{id}_{n+1}|_{k^{\text{th}} \text{ face}}] = [\pm \mathrm{id}_n] \qquad \text{(the last face)}$$

which generates $H_n(\Delta^n, \partial\Delta^n)$ by induction hypothesis. Hence id_{n+1} generates $H_{n+1}(\Delta^{n+1}, \partial\Delta^{n+1})$.

The "invariance of dimension": if nonempty $U \subseteq \mathbf{R}^m$ and $V \subseteq \mathbf{R}^n$ are homeomorphic, then $m = n$.

Mayer-Vietoris sequence wedge sum

2.3 Other topics

Manifold and the degree of a map

Euler characteristics and Betti number Euler Char: alternating sum of rank
theorem: number of k-cells,
proof: flatness of q, split exact sequence.
eg. $\chi(\mathbf{M}_g) = 2 - 2g$, $\chi(\mathbf{S}^n) = \begin{cases} 0, & n \text{ odd} \\ 2, & \text{otherwise} \end{cases}$, $\chi(\mathbf{RP}^n) = \begin{cases} 0, & n \text{ odd} \\ 1, & \text{otherwise} \end{cases}$, $\chi(\mathbf{CP}^n) = 2n$

π_1 and H_1

Simplicial approximation

3 Cohomology

3.1 Basic constructions

We dualize the chain complex by $\mathrm{Hom}(-, G)$ where G is an abelian group, so that the cochain group $C_n^* = \mathrm{Hom}(C_n, G)$ together with the coboundary map $\partial_n^* : \varphi \mapsto \varphi\partial_{n+1}$ forms a chain complex:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\ \cdots & \longleftarrow & C_{n+1}^* & \xleftarrow{\partial_n^*} & C_n^* & \xleftarrow{\partial_{n-1}^*} & C_{n-1}^* & \longleftarrow & \cdots \end{array} .$$

Construct $h : H^n(C; G) \to \mathrm{Hom}(H_n(C), G)$ by $\varphi \mapsto \overline{\varphi}_0$, where $\varphi_0 = \varphi|_{\ker \partial_n}$ and $\overline{\varphi}_0 : H_n(C) \to G$. This is appropriate since φ vanishes on $\mathrm{Im}\,\partial_{n+1}$. h is surjective by the following argument.

The short exact sequence $0 \to \ker \partial_n \xrightarrow{i} C_n \to \mathrm{Im}\,\partial_{n+1} \to 0$ splits, since $\mathrm{Im}\,\partial_{n+1} \subseteq C_n$ is free. Hence there exists $p : C_n \to \ker \partial_n$ such that $pi = \mathrm{id}_{\ker \partial_n}$. The extension $\varphi_0 p : C_n \to G$ also vanishes on $\mathrm{Im}\,\partial_{n+1}$, and $h' : \mathrm{Hom}(H_n(C), G) \xrightarrow{p} \ker \partial_n^* \to H^n(C; G)$ makes h surjective.

$$0 \longrightarrow \ker h \longrightarrow H^n(C; G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \longrightarrow 0 .$$

Now apply the zig-zag lemma to the canonical decomposition of C_n and dualize the result:

$$\begin{array}{c} \cdots \longrightarrow \mathrm{Im}\,\partial_{n+1} \xrightarrow{i_{n+1}} \ker \partial_n \longrightarrow H_n(C) \xrightarrow{\partial_n} \mathrm{Im}\,\partial_n \xrightarrow{i_n} \ker \partial_{n-1} \longrightarrow \cdots \end{array}$$

$$\overset{\text{dual}}{\Longrightarrow}$$

$$\cdots \longrightarrow \ker \partial_{n-1}^* \xrightarrow{i_{n-1}^*} \mathrm{Im}\,\partial_n^* \xrightarrow{\partial_n^*} H^n(C; G) \longrightarrow \ker \partial_n^* \xrightarrow{i_n^*} \mathrm{Im}\,\partial_{n+1}^* \longrightarrow \cdots .$$

We obtain with $\ker h = \mathrm{coker}\, i_{n-1}^* = \mathrm{Im}\,\partial_n^* / \mathrm{Im}\, i_{n-1}^* = \mathrm{Im}\,\partial_n^* / \ker \partial_{n-1}^*$,

$$0 \longrightarrow \mathrm{coker}\, i_{n-1}^* \longrightarrow H^n(C; G) \xrightarrow{h} \ker i_n^* = \mathrm{Hom}(H_n(C), G) \longrightarrow 0 .$$

Consider the dual of the free resolution $0 \to \mathrm{Im}\,\partial_n \to \ker \partial_{n-1} \to H_{n-1}(C) \to 0$ of $H_{n-1}(C)$, it happens to be that $\mathrm{Ext}(H_{n-1}(C), G) = H^1(F; G) = \mathrm{coker}\, i_{n-1}^*$. Hence $\mathrm{coker}\, i_{n-1}^*$ only depends on $H_{n-1}(C)$ and G. And since the exact sequence splits, the cohomology $H^n(C; G)$ only depends on the homology $H_{n-1}(C)$ and G. To summarize:

Theorem 3.1 (universal coefficient theorem, cohomology)**.** *The cohomology groups $H^n(C; G)$ of a chain complex C of abelian groups are determined by the split exact sequences*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C; G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \longrightarrow 0 .$$

To calculate the Ext term we have the following tools:

(i) $\mathrm{Ext}(H \oplus H', G) = \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$

(ii) $\mathrm{Ext}(H, G) = 0$ if H is free (free resolution $0 \to H \to H \to 0$)

(iii) $\mathrm{Ext}(\mathbf{Z}/n\mathbf{Z}, G) = G/nG$ (dualizing $0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \to 0$)

(iv) If H is finitely generated, then $\mathrm{Ext}(H, \mathbf{Z})$ is the torsion of H and $\mathrm{Hom}(H, \mathbf{Z})$ is the free part.

Calculation examples: torus T, M_2, Klein bottle K.

Related concepts and axioms of homology are generalized to cohomology with little or no difference. We point out the the following discrepancies:

Firstly, for $X = \bigvee_\alpha X_\alpha$, we have an isomorphism $\prod_\alpha i_\alpha^* : \widetilde{H}^n(X) \to \prod_\alpha \widetilde{H}^n(X_\alpha)$ induced by the inclusions $i_\alpha : X_\alpha \hookrightarrow X$. Since $\mathrm{Hom}(\bigoplus_\alpha A_\alpha; G) \cong \prod_\alpha \mathrm{Hom}(A_\alpha, G)$, the direct sum in homology is replaced by direct product.

We also ask why is $h : H^n(C; G) \to \mathrm{Hom}(H_n(C), G)$ an isomorphism? By the universal coefficient theorem, when $n = 0, 1$, or if we take R to be a field and consider the homology $H_n(X; F)$. But most importantly, we have the following remarkable result:

Theorem 3.2. $H^n(X^n, X^{n-1}; G) \cong \mathrm{Hom}(H_n(X^n, X^{n-1}, G))$.

Proof. Universal coefficient theorem+diagram □

3.2 The additional ring structure

The additional ring structure obtained from contravariance is useful in distinguishing spaces that the additive structure of (co)homology is incapable of doing.

The construction For ring R, define cup product $C^m(X;R) \times C^n(X;R) \xrightarrow{\smile} C^{m+n}(X;R)$ by $(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[v_0,\cdots,v_m]})\psi(\sigma|_{[v_m,\cdots,v_{m+n}]})$. Then $\partial^*(\varphi \smile \psi) = \partial^*\varphi \smile \psi + (-1)^m\varphi \smile \partial^*\psi$ implies that the cup product of cocycles is a cocycle, and the cup product of a cocycle and a coboundary is a coboundary. Hence $\smile$ induces $H^m(X;R) \times H^n(X;R) \xrightarrow{\smile} H^{m+n}(X;R)$, with the identity element $1 \in H^0(X;R)$ that sends every singular 0-simplex to 1_R.

Theorem 3.3. *$f^* : H^n(Y;R) \to H^n(X;R)$ satisfy $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$ for $f : X \to Y$.*

One can generalize to the relative version $H^m(X,A;R) \times H^n(X,B;R) \xrightarrow{\smile} H^{m+n}(X, A \cup B;R)$, by first $C^m(X,A;R) \times C^n(X,B;R) \to C^{m+n}(X,A+B;R)$ where cochains in $C^{m+n}(X,A+B;R)$ vanishes on sums of chains in A and chains in B. Then $C^{m+n}(A+B;R) \cong C^{m+n}(A \cup B;R)$ by barycentric subdivision, and $C^{m+n}(X,A+B;R) \cong C^{m+n}(X,A \cup B;R)$ by the five lemma.

Since cup product is associative and distributive, the cohomology ring $H^*(X;R) = \bigoplus_{i\geq 0} H^0(X;R)$ is a graded ring that contains a more compact description.

Theorem 3.4. *If R is commutative, then $\alpha \smile \beta = (-1)^{mn}\beta \smile \alpha$, for $|a| = m$ and $|b| = n$.*

Define a bilinear map $H^*(X;R) \times H^*(Y;R) \xrightarrow{\times} H^*(X \times Y;R)$ by $a \times b = p_1^*(a) \smile p_2^*(b)$ where p_1 and p_2 are projection from $X \times Y$ to X and Y, respectively. This induces, for commutative ring R, $H^*(X;R) \otimes_R H^*(Y;R) \xrightarrow{\mu} H^*(X \times Y;R)$ a homomorphism of R-modules $a \otimes b \mapsto a \times b$. Define multiplication $(a \otimes b)(c \otimes d) = (-1)^{|b||c|}ac \otimes bd$ in $H^*(X \times Y;R)$ to give a graded structure, and also to satisfy $\mu((a \otimes b)(c \otimes d)) = \mu(a \otimes b)\mu(c \otimes d)$.

Theorem 3.5 (Künneth formula)**.** *If $H^n(Y;R)$ is finitely-generated for all n, then the cross product μ induces $H^*(X;R) \otimes_R H^*(Y;R) \cong H^*(X \times Y;R)$.*

Proof. For CW complexes X and Y define functors $h^n(X,A) = \bigoplus_i (H^i(X,A;R) \otimes_R H^{n-i}(Y;R))$ and $k^n(X,A) = H^n(X \times Y, A \times Y;R)$. Check that h and k are (unreduced) cohomology theories:

(i) homotopy invariance is trivial;

(ii) excision is trivial if we use the alternate formulation;

(iii) existence of the long exact sequence

(iv) disjoint union

Let $\mu : h^n(X,A) \to k^n(X,A)$, then μ is natural in the sense that it commutes in the following way

That μ is an isomorphism for CW pair (X,A) follows from a reduction argument. Firstly it suffices to prove for $A = \varnothing$ by the five-lemma. Then proceed by induction on the dimension for finite-dimensional X. When $\dim X = 0$ the statement is trivial invoking axiom (iv). The inductive step X^n, which is reduced to (X^n, X^{n-1}) by the five-lemma. This is further reduced to $\coprod_\alpha (e_\alpha^n, \partial e_\alpha^n)$ by characteristic map Φ, noticing that Φ^* is an isomorphism in h and k cohomology theories. Once again by the axiom disjoint union this is reduced to $(e_\alpha^n, \partial e_\alpha^n)$, which is trivial by the five-lemma and induction hypothesis.

If X is infinite-dimensional, then it reduces to the finie-dimensional case by the telescope argument. □

Examples

(i) Let M_2 be an orientable surface of genus 2, then $H^n(M_2;\mathbf{Z}) \cong \mathrm{Hom}(H_n(M_2),\mathbf{Z})$ by cellular cohomology, and $H^0(M_2;\mathbf{Z}) = H^2(M_2;\mathbf{Z}) = \mathbf{Z}$ and $H^1(M_2;\mathbf{Z}) = \mathbf{Z}^4$. Let $\{\alpha_i\}_{1\le i\le 4}$ be the basis of $H^1(M_2;\mathbf{Z})$, then α_i sends a_i to $\mathbf{1}_R$ and other generators to 0. One can check that the vanishing condition $\partial_1^*\alpha_i = 0$ makes α_i a traversing line between two a_i. Applying cup product we find that $\alpha_1 \smile \alpha_2 = 1$ on $[0,2,1]$ and $\alpha_2 \smile \alpha_1 = 1$ on the right adjacent one. This 1 is the generator of $H^2(M_2;\mathbf{Z})$.

(ii) Let T^2 be a torus, then $H^0(T^2;\mathbf{Z}) = H^2(T^2;\mathbf{Z}) = \mathbf{Z}$ and $H^1(T^2;\mathbf{Z}) = \mathbf{Z}^2 = (\alpha,\beta)$. On the Δ-complex structure of T^2 we see that $\alpha \smile \beta = -(\beta \smile \alpha)$ generates $H^2(T^2;\mathbf{Z})$. Hence $H^*(T^2;\mathbf{Z}) = \mathbf{Z}[\alpha,\beta]/(\alpha^2,\beta^2,\alpha\beta = -\beta\alpha) = \Lambda_{\mathbf{Z}}[\alpha,\beta]$, a result also obtainable by the Künneth formula $H^*(\mathbf{S}^1\times\mathbf{S}^1;\mathbf{Z}) = \Lambda_{\mathbf{Z}}[\alpha]\otimes_{\mathbf{Z}}\Lambda_{\mathbf{Z}}[\beta] = \Lambda_{\mathbf{Z}}[\alpha,\beta]$, since $|\alpha| = |\beta| = 1$ is odd.

In general, since $H^*(\mathbf{S}^n;\mathbf{Z}) = \Lambda_{\mathbf{Z}}[\alpha] = \mathbf{Z}[\alpha]/(\alpha^2)$ where $|\alpha| = n$, by the Künneth formula $H^*(\mathbf{S}^{2m}\times\mathbf{S}^{2n+1};\mathbf{Z}) = \mathbf{Z}[\alpha]/(\alpha^2)\otimes_{\mathbf{Z}}\Lambda_{\mathbf{Z}}[\beta]$ where $|\alpha| = 2m$ and $|\beta| = 2n+1$. The distinction of parity comes from the commutativity criteria.[1]

(iii) Let K be a Klein bottle, $\partial_1^* : \alpha \mapsto \eta + \zeta$ since $\partial_1^*\alpha(U) = \partial_1^*\alpha(L) = 1$, and similarly, $\beta \mapsto \eta - \zeta$ and $\gamma \mapsto \zeta - \eta$. Hence $\mathrm{Im}\,\partial_1^* = (\eta+\zeta, \eta-\zeta, \zeta-\eta) = (\eta+\zeta)$, taking $R = \mathbf{Z}_2$, and $\ker\partial_1^* = (\alpha+\beta, \alpha+\gamma)$. Since $\partial_0^*\mu$ on any of a,b,c is 0, $\mathrm{Im}\,\partial_0^* = 0$. In summary, $H^2(K;\mathbf{Z}_2) = (\eta,\zeta)/(\eta+\zeta) = (\eta)$ and $H^1(K;\mathbf{Z}_2) = (\alpha+\beta,\alpha+\gamma)$. Finally let $x = \alpha+\beta$ and $y = \beta+\gamma$, then $x\smile x(U) = x \smile y(U) = 1$ and all other terms are 0. Hence $x\smile x = x\smile y$ acting on U gives η. We obtain $H^*(K,\mathbf{Z}_2) = \mathbf{Z}_2[x,y]/(x^3,y^3,xy,x^2-y^2)$ with basis $1, x, x^2, y$.

(iv) Hatcher Example 3.16 says that the exterior algebra $\Lambda_R[\alpha_1,\cdots,\alpha_n]$ is the graded tensor product over R of $\Lambda_R[\alpha_i]$ where all α_i has odd degree/dimension. I want to make sure that the antisymmetry $\alpha_i\alpha_j = -\alpha_j\alpha_i$ in $\Lambda_R[\alpha_1,\cdots,\alpha_n]$ is guaranteed by the odd degree like this:

$$\alpha_i\alpha_j = (\mathbf{1}_{\Lambda_R[\alpha_i]}\otimes\alpha_i)(\alpha_j\otimes\mathbf{1}_{\Lambda_R[\alpha_i]}) = (-1)^{1\cdot 1}\alpha_j\alpha_i$$

[1] When $|\alpha|$ and $|\beta|$ is odd, $\Lambda_R[\alpha,\beta]$ guarantees $\alpha\beta = -\beta\alpha$ but $\mathbf{Z}[\alpha]/(\alpha^2)\otimes_{\mathbf{Z}}\mathbf{Z}[\beta]/(\beta^2)$ does not. The distinction becomes irrelevant if $\mathrm{char}\,R = 2$.

3.3 Application to manifolds

An n-manifold is a second countable Hausdorff space locally homeomorphic to $\mathbf{R}^n$.

Orientation Let X be a manifold. A generator $\mu_x \in H_n(X, X \setminus \{x\}) \cong \mathbf{Z}$ is a local orientation at $x \in X$. $(\widetilde{X}, \pi)$ is a covering space where $\widetilde{X} = \{(x, \mu_x)\}$ and π is the natural projection. Topologize $\widetilde{X}$: for open $U \subseteq X$ homeomorphic to $\mathbf{R}^n$, $\pi^{-1}(U)$ consists of (x, μ_x) such that $x \in U$ and μ_x is the image of $H_n(X, X \setminus U) \cong H_n(X, X \setminus \{x\})$. X is orientable if there is a section $s : X \to \widetilde{X}$ with $\pi s = \mathbf{1}_X$. With this topology the continuity of s implies the compatibility of orientation.

For a connected manifold X, it is orientable if and only if $\widetilde{X}$ has two connected components $X \sqcup X$. Hence if X is simply connected, then it is orientable since it does not have subgroups of index 2. Also the fact that $\mathbf{S}^2$ covers $\mathbf{R}P^2$ tells us that $\mathbf{R}P^2$ is non-orientable.

We also have a more general $\mathbf{Z}$-cover $\widetilde{X}_{\mathbf{Z}} = \bigcup_{k \geq 0} \widetilde{X}_k$ where $\widetilde{X}_0 \cong X$ and $\widetilde{X}_k \cong \widetilde{X}$ for $k > 0$. More generally, for a commutative ring R choosing units $\mu_x \in H_n(X, X \setminus \{x\}; R)$, we have a covering space $\widetilde{X}_R = \bigcup_{r \in R} \widetilde{X}_r$ where $\widetilde{X}_r \cong X$ if $r = 0$ or $2r = 0$ and $\widetilde{X}_r \cong \widetilde{X}$ otherwise. R-orientability is the existence of a section $s : X \to \widetilde{X}_R$ with $s(x)$ is a unit of $H_n(X, X \setminus \{x\}; R) \cong R$.

If R has a unit of order 2, then every space X is R-orientable by taking the identity map; otherwise, a space is R-orientable if and only if it is $\mathbf{Z}$-orientable, in terms of a double cover $\widetilde{X}$.

Theorem 3.6. *If X is a compact connected n-manifold, then:*

(i) If X is R-orientable, then $H_n(X; R) \cong H_n(X, X \setminus \{x\}; R)$;

(ii) otherwise $H_n(X; R) \to H_n(X, X \setminus \{x\}; R)$ is injective and has image $\{r \in R; 2r = 0\}$.

(iii) $H_k(X; R) = 0$ for $k > n$. (Non-trivial since not all manifolds have triangulation.)

Proof.

$$\begin{array}{ccc} H_n(X;R) & \dashrightarrow & H_n(X, X \setminus \{x\}; R) \\ & \searrow \quad \swarrow_{\cong} & \\ & \Gamma_R(X) & \end{array}$$

□

Theorem 3.7 (Poincaré Duality)**.** *Let X be a R-orientable n-manifold with fundamental class μ, then $H_c^k(X; R) \cong H_{n-k}(X; R)$ for all k by $D_X : \alpha \mapsto \mu \frown \alpha$.*

Proof. We give two proofs, one more geometric and one more algebraic.

Second proof. We appeal to two inductive facts:

(i) Let $X = U \cup V$ where U, V are open, then if $D_U, D_V, D_{U \cap V}$ are isomorphisms, so is D_X.

(ii) Let $X = \bigcup_{\alpha \in I} U_\alpha$ where U_α are open sets, then if D_{U_α} are isomorphisms, so is D_X

(i) follows from the five-lemma and the the commutativity (up to a $\pm$ sign) of the following diagram:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H_c^k(U \cap V) & \longrightarrow & H_c^k(U) \oplus H_c^k(V) & \longrightarrow & H_c^k(X) & \xrightarrow{\delta} & H_c^{k+1}(U \cap V) & \to \cdots \\ & \downarrow D_{U \cap V} & & \downarrow D_U \oplus -D_V & & \downarrow D_X & & \downarrow D_{U \cap V} & \\ \cdots \longrightarrow & H_{n-k}(U \cap V) & \longrightarrow & H_{n-k}(U) \oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(X) & \xrightarrow{\partial} & H_{n-k}(U \cap V) & \longrightarrow \cdots \end{array}.$$

All the maps are natural except for the connecting homomorphisms δ and ∂ in the Mayer-Vietoris sequences. The commutativity of the last square is obtained by passing the limit over compact $K \subseteq U$ and $L \subseteq V$ in the following diagram, noting that the direct limit preserves exactness:

$$\begin{array}{ccc} H^k(X|K \cup L) & \xrightarrow{\delta} & H^{k+1}(X|K \cap L) \cong H^{k+1}(U \cap V|K \cap L) \\ \downarrow{\scriptstyle \mu_{K\cup L}\frown} & & \downarrow{\scriptstyle \mu_{K\cap L}\frown} \\ H_{n-k}(X) & \xrightarrow{\partial} & H_{n-k-1}(U \cap V) \end{array}.$$

Indeed, blah

(ii) follow from properties of direct limit:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_c^k(U_\alpha) \cong H_c^k(X|K_\alpha) & \xrightarrow{D_{U_\alpha}} & H_{n-k}(U\alpha) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varinjlim_\alpha} & & \downarrow{\scriptstyle \varinjlim_\alpha} & & \\ 0 & \longrightarrow & \varinjlim_\alpha H_c^k(X|K_\alpha) \cong H_c^k(X) & \xrightarrow{D_X} & H_{n-k}(X) & \longrightarrow & 0 \end{array}$$

where K_α ranges over compact subsets of U_α.

First let $X = \mathbf{R}^n$, then $H_c^k(\mathbf{R}^n; R) \cong R \cong H^k(\Delta^n, \partial\Delta^n; R)$, only when $k = n$. Choose a generator $\mu \in H_n(\Delta^n, \partial\Delta^n; R)$ which is an n-simplex, then $D_X(\alpha) = \mu \frown \alpha = \alpha(\mu)$ is a generator in $H_0(X; R) \cong R$, for a generator $\alpha \in H^n(\Delta^n, \partial\Delta^n; R) = \mathrm{Hom}(H_n(\Delta^n, \partial\Delta^n), R)$.

By (i) D_X is an isomorphism for X a finite union of open convex sets each homeomorphic to $\mathbf{R}^n$. By (ii) the statement also holds for any open $X \subseteq \mathbf{R}^n$, since X is a countable union of such.

Repeat the procedure as above, then the statement holds for X a countable union of open sets each homeomorphic to $\mathbf{R}^n$. For uncountable union use (ii) and Zorn's lemma. □

If X is compact, then $H_c^k(X; R) = H^k(X; R)$.

As a corollary, the Euler characteristic of an odd dimensional compact manifold is zero. The orientable case is easy; otherwise let $R = \mathbf{Z}_2$ and conclude that $\dim H_i(X; \mathbf{Z}_2) = \mathrm{rank}\, H_i(X; \mathbf{Z})$ by analyzing torsion using the universal coefficient theorem for cohomology.

4 Homotopy Theory

Higher homotopy groups π_n is a covariant functor from **Top*** to **Set** ($n = 0$), **Grp** ($n = 1$), and **Ab** ($n \geq 2$). Verify the homotopy invariance.

Construction For a based space (X, x_0), define $\pi_n(X, x_0)$ to be the set of homotopy classes of maps $f : (I^n, \partial I^n) \to (X, x_0)$ under "fix-end" ($f_t(\partial I^n) = x_0$ for all t) homotopies. Define sum operation in $\pi_n(X, x_0)$ for $n \geq 2$

$$(f+g)(s_1, \cdots, s_n) = \begin{cases} f(2s_1, s_2, \cdots, s_n), & s_1 \in [0, 1/2] \\ f(2s_1 - 1, s_2, \cdots, s_n), & s_1 \in [1/2, 1] \end{cases},$$

which makes $\pi_n(X, x_0)$ into an abelian group. Alternatively, one think $\pi_n(X, x_0)$ as the set of homotopy classes of maps $f : (\mathbf{S}^n, s_0) \to (X, x_0)$ under homotopies of the same form. If X is path-connected, then the base point is irrelevant.

Theorem 4.1. *A covering map $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ induces $\pi_n(\widetilde{X}, \widetilde{x}_0) \cong \pi_n(X, x_0)$ for $n \geq 2$.*

Proof. For $f : (\mathbf{S}^n, s_0) \to (X, x_0)$, as $\mathbf{S}^n$ is connected and locally connected and $\pi_1(\mathbf{S}^n) = 0$, by the lifting criterion there exists $\widetilde{f}$ such that $p_*([\widetilde{f}]) = [f]$. On the other hand, if $g \in \ker p_*$, then $p(g) \simeq 0 \in \pi_n(X, x_0)$, which lifts to a constant map in $\pi_n(\widetilde{X}, \widetilde{x}_0)$. Hence $g \simeq 0 \in \pi_n(\widetilde{X}, \widetilde{x}_0)$. □

As a result, since $\mathbf{T}^n$ is covered by $\mathbf{R}^n$ (identifying all the lattice points in $\mathbf{R}^n$) which is contractible, $\pi_k(\mathbf{T}^n) = 0$ for $k \geq 2$.

Theorem 4.2. *For each path-connected space X_α, there are $\pi_n(\coprod_\alpha X_\alpha) \cong \prod_\alpha \pi_n(X_\alpha)$ for all n, contrasting to homology groups.*

Define relative homotopy group $\pi_n(X, A, x_0) = [(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)]$ (or alternatively $[(\mathbf{D}^n, \mathbf{S}^{n-1}, s_0) \to (X, A, x_0)]$) with homotopies of the same form, where $J^{n-1} = \partial I^n - I^{n-1}$, I^{n-1} being the face with last coordinate 0. A map $(\mathbf{D}^n, \mathbf{S}^{n-1}, s_0) \to (X, A, x_0) = 0 \in \pi_n(X, A, x_0)$ if and only if it is homotopic $\operatorname{rel} \mathbf{S}^{n-1}$ to a map with images in A.

Theorem 4.3. *Let i and j be inclusions and ∂ be the restriction of $f \in \pi_n(X, A, x_0)$ to I^{n-1} (or $\mathbf{S}^{n-1}$), then the following sequence is exact:*

$$\cdots \xrightarrow{i_*} \pi_n(A, B, x_0) \xrightarrow{j_*} \pi_n(X, B, x_0) \xrightarrow{\partial} \pi_{n-1}(X, A, x_0) \xrightarrow{i_*} \cdots.$$

On CW complexes Whitehead, extension, compression, cellular, CW

A CW pair (X, A) is n-connected if cells in $X \setminus A$ have dimensions greater than n. In particular, $\pi_i(X^n) = \pi_i(X)$ is an isomorphism for $i < n$ and a surjection for $i = n$.

Homotopy excision and Eilenberg-MacLane spaces Excision for homotopy only works for CW complexes in a certain range of dimensions, as opposed to homology excision.

Theorem 4.4 (excision)**.** *Let $X = A \cup B$ be a CW complex where A and B are subcomplexes and $C = A \cap B$ is nonempty and connected. If (A, C) is m-connected and (B, C) is n-connected, then $\pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism for $i < m + n$ and a surjection for $i = m + n$.*

Two useful consequences of homotopy excision are:

Theorem 4.5 (Freudenthal suspension theorem)**.** *If CW complex X is $(n-1)$-connected, then the suspension $\pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$.*

Proof. Since cones are contractible, from the long exact sequence of pairs we obtain $\pi_{i+1}(SX) \cong \pi_{i+1}(SX, C_-X)$ and $\pi_{i+1}(C_+X, X) \cong \pi_i(X)$. By the same exact sequence (C_+X, X) and (C_-X, X) are n-connected if X is $(n-1)$-connected. The statement follows from homotopy excision. □

Theorem 4.6. *If a CW pair (X, A) is m-connected and A is n-connected, then $\pi_i(X, A) \to \pi_i(X/A)$ is an isomorphism for $i < m + n + 1$ and a surjection for $i = m + n + 1$.*

Consequently we can calculate $\pi_n(\mathbf{S}^n)$ from the sequence $\mathbf{Z} = \pi_1(\mathbf{S}^1) \twoheadrightarrow \pi_2(\mathbf{S}^2) \to \cdots$ where all maps after the first are isomorphisms. Since maps $\mathbf{S}^n \to \mathbf{S}^n$ of arbitrary degrees exist and degree is homotopy invariant, $\pi_2(\mathbf{S}^2) \to \mathbf{Z}$ is surjective. Hence $\pi_n(\mathbf{S}^n) = \mathbf{Z}$ for all n.

Then we can calculate $\pi_n(\bigvee_\alpha \mathbf{S}^n)$.

Finally, the most important calculation is for $X = (\bigvee_\alpha \mathbf{S}^n) \cup_\beta e_\beta^{n+1}$ where e_β^{n+1} is attached by basepoint-preserving inclusions $\varphi_\beta : \mathbf{S}^n \to \bigvee_\alpha \mathbf{S}^n$. By cellular approximation $\pi_i(X) = 0$ for $i < n$. Consider the long exact sequence $\pi_{n+1}(X, \bigvee_\alpha \mathbf{S}^n) \xrightarrow{\partial} \pi_n(\bigvee_\alpha \mathbf{S}^n) \to \pi_n(X) \to \pi_n(X, \bigvee_\alpha \mathbf{S}^n)$ where $\pi_{n+1}(X, \bigvee_\alpha \mathbf{S}^n)$ collapses to $\pi_{n+1}(\bigvee_\alpha \mathbf{S}_\beta^{n+1})$ and $\pi_n(X, \bigvee_\alpha \mathbf{S}^n) = 0$ by cellular approximation. $\pi_{n+1}(\bigvee_\alpha \mathbf{S}_\beta^{n+1})$ is free abelian with basis φ_β, which is sent to $[\varphi_\beta]$ by ∂. Hence $\pi_n(X) = \oplus_\alpha \mathbf{Z}/[\varphi_\beta]$. Any subgroup can be realized as such.

The Eilenberg-MacLane space $X = K(G, n)$ is the space with the only nontrivial homotopy group $\pi_n(X) = G$. The calculation above together with Postnikov tower gives an explicit construction of $K(G, n)$ with CW structure. Examples of

Theorem 4.7 (Hurewicz theorem)**.** *If (X, A) is $(n-2)$-connected for $n \geq 2$ and $A \neq \varnothing$ connected, then $H_i(X, A) = 0$ for $i < n$ and $h : \pi_n(X, A) \to H_n(X, A)$ by $[f] \mapsto f_*(\alpha)$ is an isomorphism where α is a generator of $H_n(\mathbf{D}^n, \mathbf{S}^{n-1}) = \mathbf{Z}$.*

Fibration A map $p : E \to B$ have homotopy lifting property with respect to X if for a homotopy $f_t : X \to B$ and a map $\widetilde{f_0} : X \to E$ lifting f_0, there exists a homotopy $\widetilde{f_t} : X \to E$ lifting f_t. If instead of X there is a pair (X, A), then $f_t : X \to B$ lifts to $\widetilde{f_t} : X \to E$ which starts at a given lift $\widetilde{f_0}$ and extends a given lift $\widetilde{g_t} : A \to E$.

$$\begin{array}{ccc} (X \times \{0\}) \cup (A \times I) & \xrightarrow{\widetilde{f_0} \cup \widetilde{g_t}} & E \\ \big\downarrow & \overset{\widetilde{f_t}}{\nearrow} & \big\downarrow p \\ (X, A) & \xrightarrow[f_t]{} & B \end{array}$$

Fibers of a map $p : E \to B$ are elements $p^{-1}(b) \subseteq E$. The map p is a fibration if it has homotopy lifting property with respect to all spaces X; it is a Serre fibration if it has homotopy lifting property for disks $\mathbf{D}^k$, or equivalently for pairs $(\mathbf{D}^k, \partial\mathbf{D}^k)$ by deformation retraction, where $k \geq 0$.

Theorem 4.8. *Let $p : E \to B$ be a Serre fibration, then for a fixed $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$, the map $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0, b_0)$ is an isomorphism for $n \geq 1$.*

Proof. blah □

A fiber bundle over E is $F \to E \xrightarrow{p} B$ with fiber F such that every $p \in B$ has a neighborhood U for which the local trivialization $h : p^{-1}(U) \to U \times F$ is a homeomorphism. Fiber bundles have a local product structure.

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{h} & U \times F \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & U & \end{array}$$

Examples include Möbius band, and Hopf fibration.

Theorem 4.9. *A fibre bundle $p : E \to B$ has homotopy lifting property with respect to all CW pairs.*

[Hurewicz] If a pair (X, A) is $(n-1)$-connected for $n > 1$, then $h : \pi_i(X, A) \to H_i(X, A)$ by $[f] \mapsto f_*(\alpha)$ is an isomorphism for $i \leq n$ with α being a generator of $H_n(\mathbf{D}^n, \mathbf{S}^{n-1}) = \mathbf{Z}$.
[Whitehead] For a map $f : X \to Y$ between 1-connected spaces, the following are equivalent for $n \geq 1$:

(i) The induced $\pi_i(f) : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i < n$ and an epimorphism for $i = n$;

(ii) the induced $H_i(f) : H_i(X) \to H_i(Y)$ is an isomorphism for $i < n$ and and epimorphism for $i = n$.

Let $\mathfrak{C}$ be a Serre class and X be 1-connected. Then $\pi_i(X) \in \mathfrak{C}$ for all i if and only if $H_i(X) \in \mathfrak{C}$ for all i.
[mod-$\mathfrak{C}$ Hurewicz] Let $\mathfrak{C}$ be a Serre class and (X, A) be 1-connected. If $\pi_i(X, A) \in \mathfrak{C}$ for $i < n$, then $H_i(X, A) \in \mathfrak{C}$ for $i < n$ and the Hurewicz map $h : \pi_i(X, A) \to H_i(X, A)$ is a $\mathfrak{C}$-isomorphism for $i \leq n$.